Synthesis for Structure Rewriting Systems

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MFCS

High Tatras, 2009

Structure Rewriting History

Relational Structures and Dynamics of Certain Discrete Systems

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Since then …

- Theory of **graph grammars**, **tree decompositions**, connections to **MSO**
- Applications to **software engineering and verification**

Structure Rewriting Rules

Relational Structures and Embeddings

$$\sigma \ : \quad \mathfrak{A} = (A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \dots, R_k^{\mathfrak{A}}) \quad \to \quad (B, R_1^{\mathfrak{B}}, R_2^{\mathfrak{B}}, \dots, R_k^{\mathfrak{B}}) = \mathfrak{B}$$

Embedding: σ is **injective** and $R_i^{\mathfrak{A}}(a_1, \dots, a_{r_i}) \Leftrightarrow R_i^{\mathfrak{B}}(\sigma(a_1), \dots, \sigma(a_{r_i}))$

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Rewriting Definition

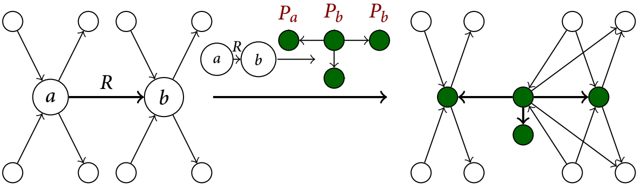
$\mathfrak{B} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$ iff $B = (A \setminus \sigma(L))\dot\cup R$ and,

for $M = \{(r, a) \mid a = \sigma(l), r \in P_l^{\mathfrak{R}}$ for some $l \in L\} \cup \{(a, a) \mid a \in A\}$,

$$(b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{B}} \;\Leftrightarrow\; (b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{R}} \textbf{ or } (b_1 M \times \ldots \times b_{r_i} M) \cap R_i^{\mathfrak{A}} \neq \varnothing.$$

(in the second case at least one $b_j \notin \mathfrak{A}$)

Rewriting Example



Structure Rewriting Games

Game arena is a **directed graph** with:

- vertices partitioned into positions of **Player 0** and **Player 1**
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- **Existential**: $\mathfrak{A}_{\text{next}} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$, the player chooses the embedding σ
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Winning condition:

- L_μ (or temporal) formula ψ with **MSO sentences** for predicates, or
- MSO formula φ to be evaluated on the **limit** of the play
  Limit of $\mathfrak{A}_0\mathfrak{A}_1\mathfrak{A}_2 \ldots \ = \ \left(\bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} A_i,\ \bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} R^{\mathfrak{A}_i}\right)$

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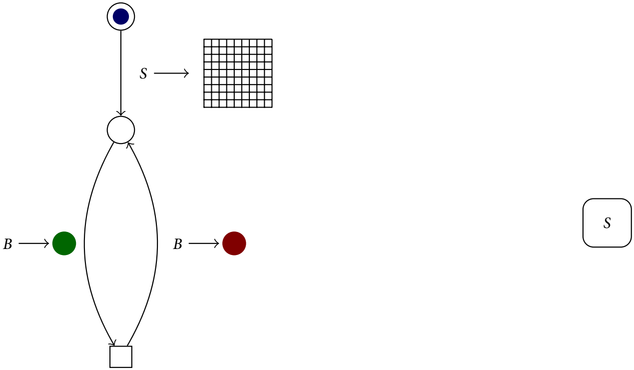
Motivation: many questions are **naturally defined as such games**:
constraint satisfaction, model checking, graph measures, games people play

Example Game: Connect–5



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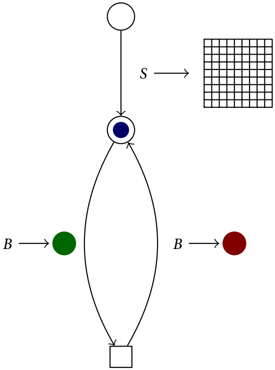

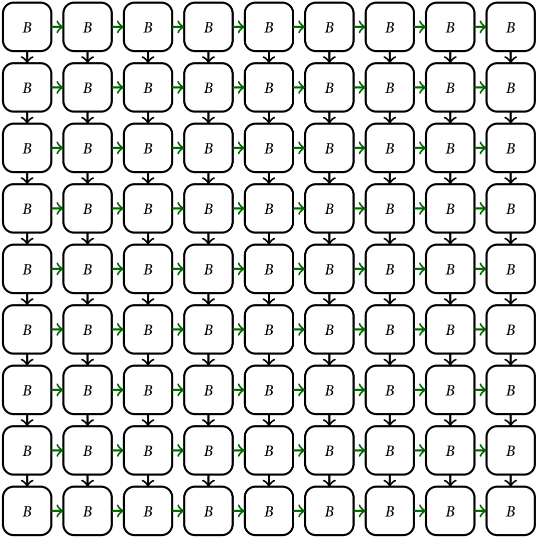

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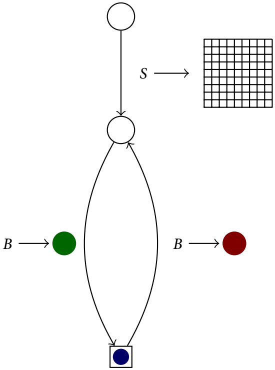

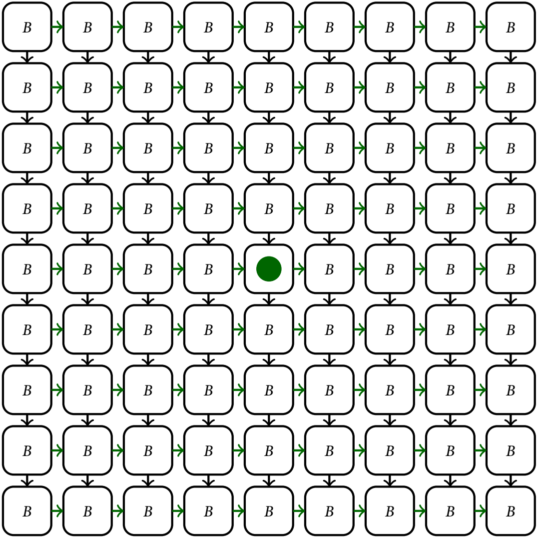

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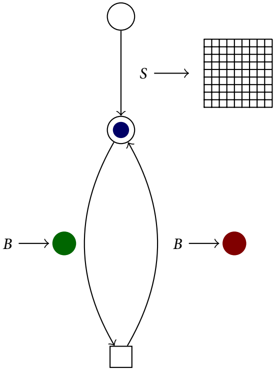

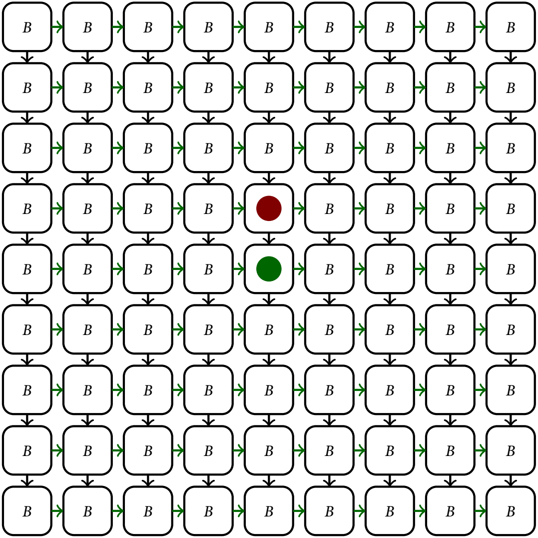

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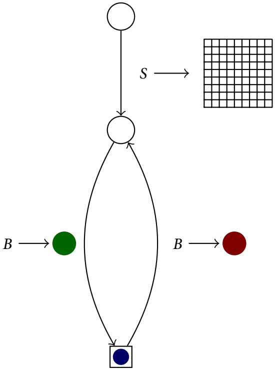

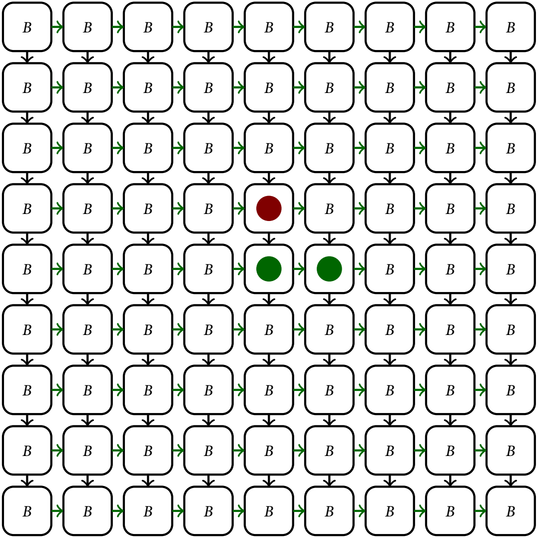

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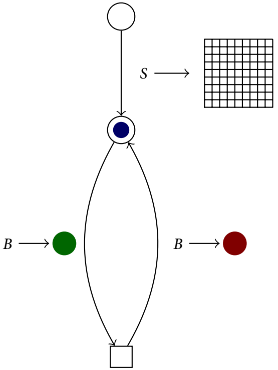
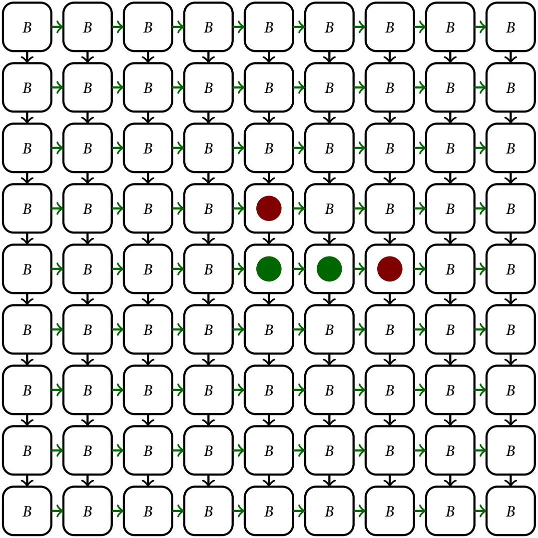


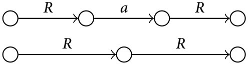


$$\exists x_1 \dots x_5 \Big(\bigwedge_{1\le i\le 5} G(x_i) \wedge \Big(\bigwedge_{1\le i\le 5} R(x_i, x_{i+1}) \vee \bigwedge_{1\le i\le 5} C(x_i, x_{i+1}) \vee \bigwedge_{1\le i\le 5} \exists y \big(R(x_i, y) \wedge C(y, x_{i+1})\big) \vee \bigwedge_{1\le i\le 5} \exists y \big(R(x_i, y) \wedge C(x_{i+1}, y)\big)\Big)\Big)$$

Simple Structure Rewriting

Separated Structures: no element is in two non-terminal relations
(Courcelle, Engelfriet, Rozenberg, 1991)

Separated:
Not Separated:



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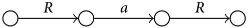
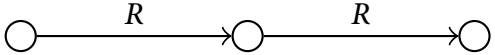
Separated: 

Not Separated: 

Simple Rule $\mathfrak{L} \to \mathfrak{R}$: $\mathfrak{R}$ is **separated** and $\mathfrak{L}$ is a **single tuple in relation**

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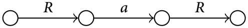

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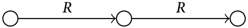

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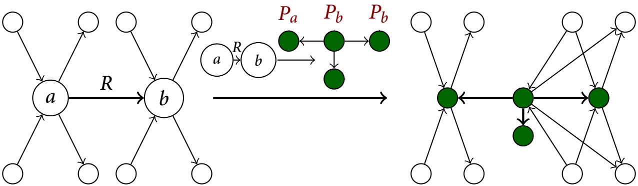

Main Result

Logics

- $\mathrm{L}_\mu[\mathrm{MSO}]$: Temporal properties expressed in L_μ (subsumes LTL) with properties of structures (states) expressed in MSO
- lim MSO: Property of the limit structure expressed in MSO

Theorem

- *Let R be a* ***finite*** *set of* ***(universal) simple structure rewriting rules,***
- *and φ be an $\mathrm{L}_\mu[\mathrm{MSO}]$ or lim MSO formula.*

Then the set $\{\pi \in R^\omega : (\lim) S(\pi) \vDash \varphi\}$ is ***ω-regular****.*

Corollary

Establishing the winner of (universal) finite simple rewriting games is decidable.

Why Universal Rewriting?

Simple Rewriting: Universal vs. Existential

- **Universal** is arguably **less natural** than the **Existential**
- **Graph grammars** (one player) are defined in the **Existential** way
- **Generated structures** have bounded clique-width **in both cases**

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Establishing the winner in existential games is undecidable:
Simulate **active context-free games** (thanks to **Anca Muscholl**)

Active Context-Free Games

- Played on **a word** (letters $\rightsquigarrow$ predicates)
- **Juliet** selects **a position** in the word
- **Romeo** selects **a CFG rule** to apply
- **Winner:** Juliet wins if a word in **a regular language** L is reached

Proof: Intuition behind Simple Rewriting

Simple Rewriting ignoring Terminal Relations

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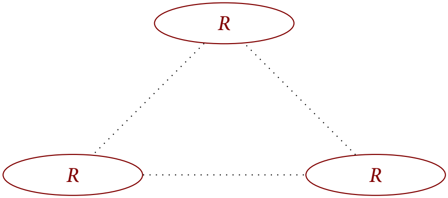

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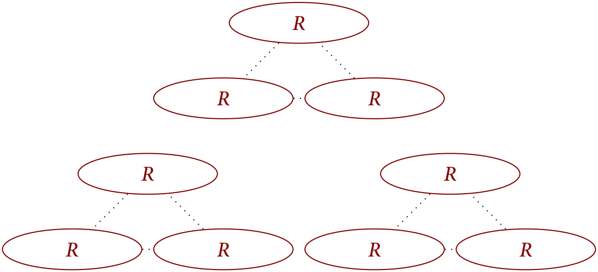

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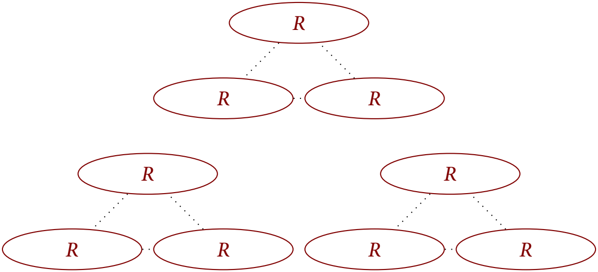

MSO is compositional:

$$\mathrm{Th}^k(\mathfrak{A} \oplus^{\text{connect}} \mathfrak{B}) = \mathrm{Th}^k(\mathfrak{A}) \oplus^{\text{connect}} \mathrm{Th}^k(\mathfrak{B})$$

Not compositional: e.g. $|P| = |Q|$ (in SO, MSO_2 using Hamilton cycle)

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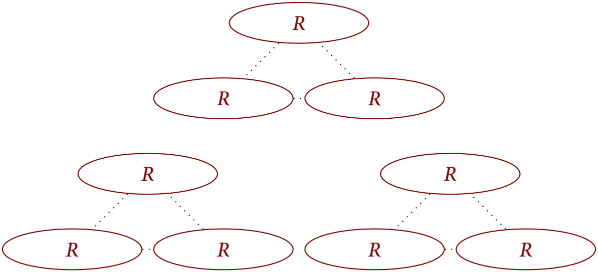

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Proof formally: through MSO interpretation in the binary tree

Proof: Interpreting a Structure in a Tree

Description of how to build $\mathfrak{A}$ is a tree $\mathcal{T}(\mathfrak{A})$ with:

- **Leafs** of **different colours** $1 \ldots k$
- $\oplus$ representing **disjoint sum**
- $i \leftarrow j$ to **change colour** of **all** nodes from i to j
- $e(i, j)$ to **add all pairs** of (i, j)-coloured nodes to e

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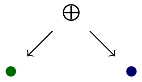

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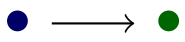

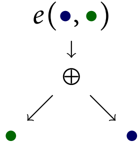

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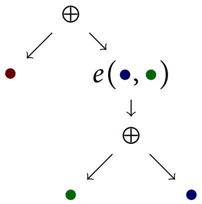

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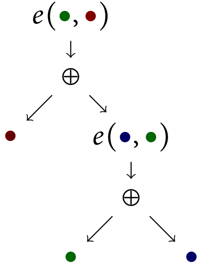

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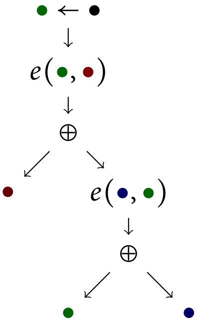

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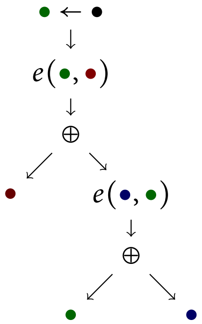

Theorem:
For every k there is an MSO-to-MSO **interpretation** $\mathcal{I}$ such that for all structures $\mathfrak{A}$ of **clique-width** $\leq k$ holds $\mathcal{I}(\mathcal{T}(\mathfrak{A})) \cong \mathfrak{A}$.

Proof: Simple Rewriting in the Tree

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$\downarrow$

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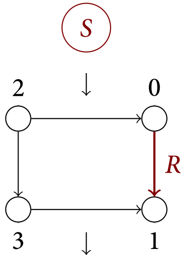

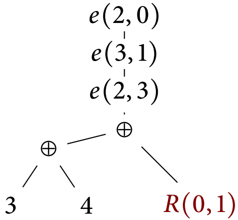

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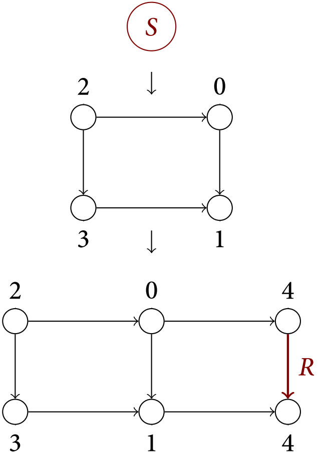

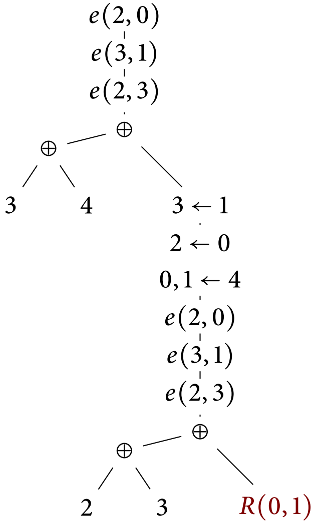

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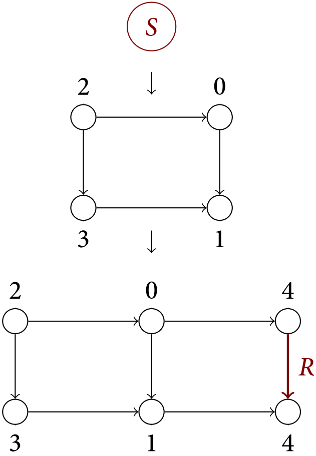
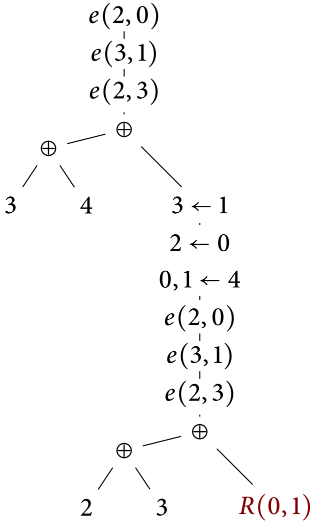
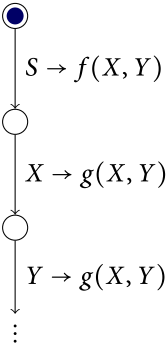


MSO-to-MSO **interpretation:** $\varphi \to \psi$



Proof: From Tree to Alternating Word Automata



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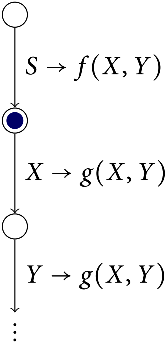

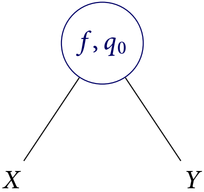

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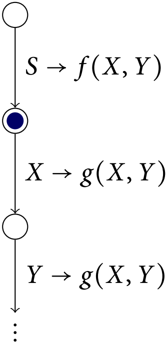

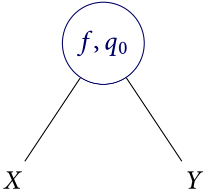

existential: pick transition

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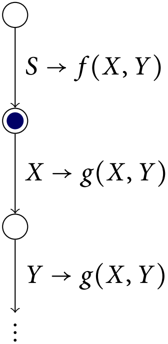

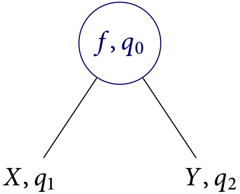

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$f, q_0 \to (q_1, q_2)$

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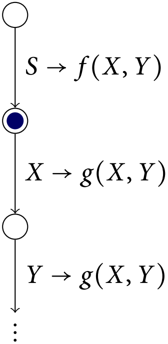

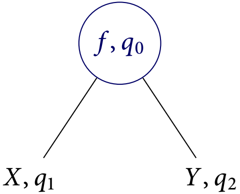

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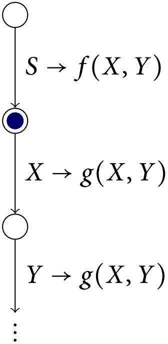

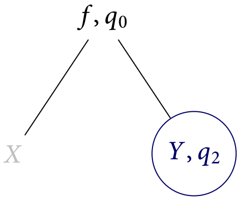

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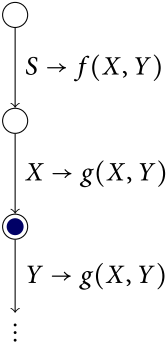

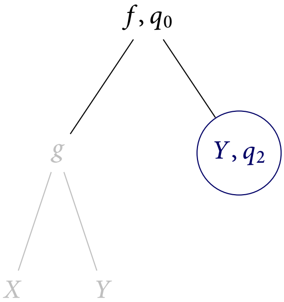

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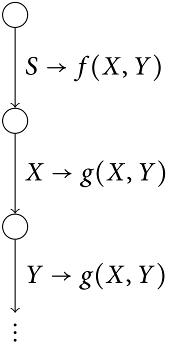

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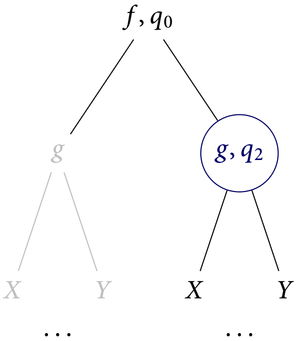

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ignore

Outlook

Many questions are naturally defined as structure rewriting games.

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Important in practice, e.g. for software verification (separation logic)

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Thank You