Analyzing Structure Rewriting Systems

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RWTH Aachen

AlgoSyn

Aachen, 2009

Theory meets Practice

Theory meets Practice

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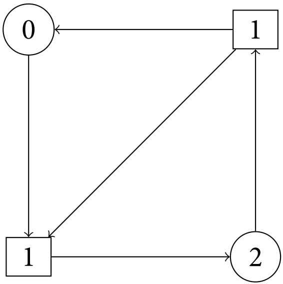

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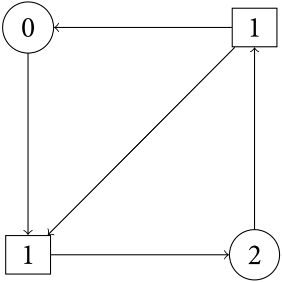

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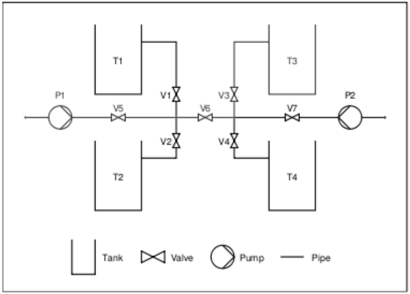

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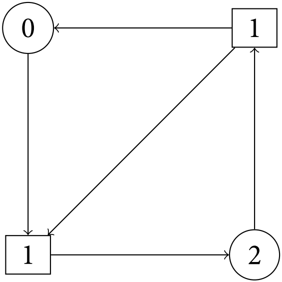

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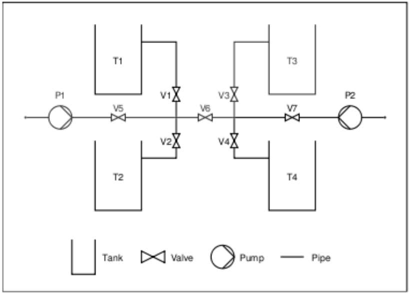

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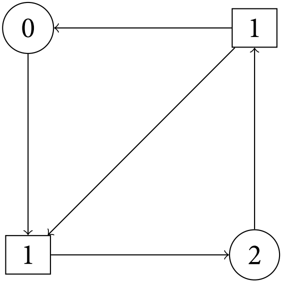

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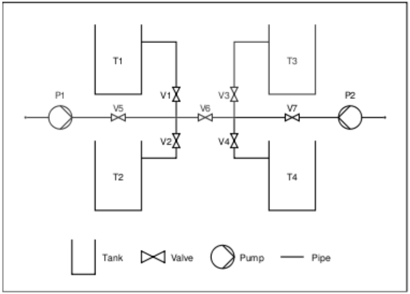

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Can we model states by arbitrary relational structures?

Structure Rewriting Rules

Relational Structures and Embeddings

$$\sigma \; : \quad \mathfrak{A} = (A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \ldots, R_k^{\mathfrak{A}}) \quad \to \quad (B, R_1^{\mathfrak{B}}, R_2^{\mathfrak{B}}, \ldots, R_k^{\mathfrak{B}}) = \mathfrak{B}$$

Embedding: σ is **injective** and $R_i^{\mathfrak{A}}(a_1, \ldots, a_{r_i}) \Leftrightarrow R_i^{\mathfrak{B}}(\sigma(a_1), \ldots, \sigma(a_{r_i}))$

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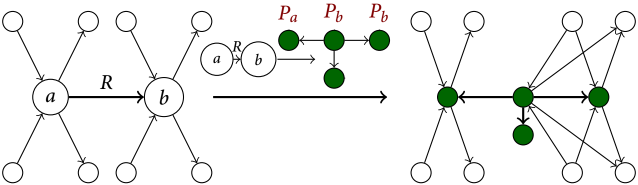
Rewriting Definition

$\mathfrak{B} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$ iff $B = (A \setminus \sigma(L)) \dot{\cup} R$ and,
for $M = \{(r, a) \mid a = \sigma(l), r \in P_l^{\mathfrak{R}} \text{ for some } l \in L\} \cup \{(a, a) \mid a \in A\}$,

$$(b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{B}} \;\Leftrightarrow\; (b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{R}} \textbf{ or } (b_1 M \times \ldots \times b_{r_i} M) \cap R_i^{\mathfrak{A}} \neq \emptyset.$$

(in the second case at least one $b_j \notin \mathfrak{A}$)

Rewriting Example



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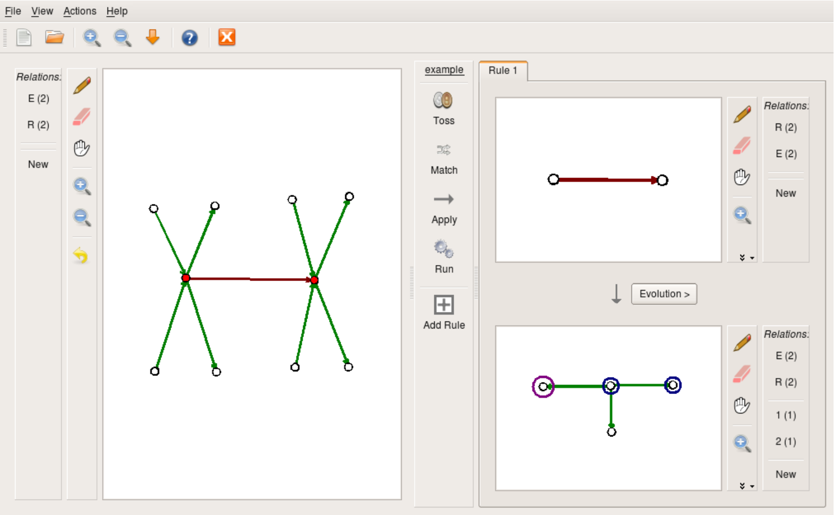

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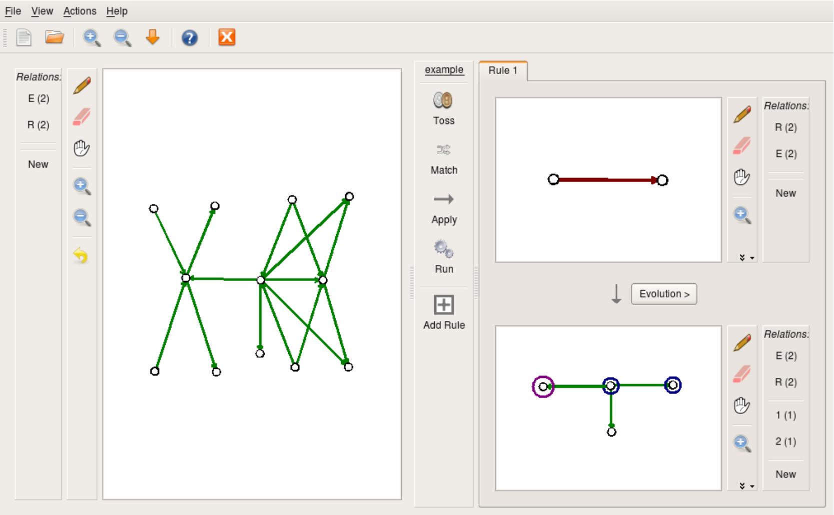
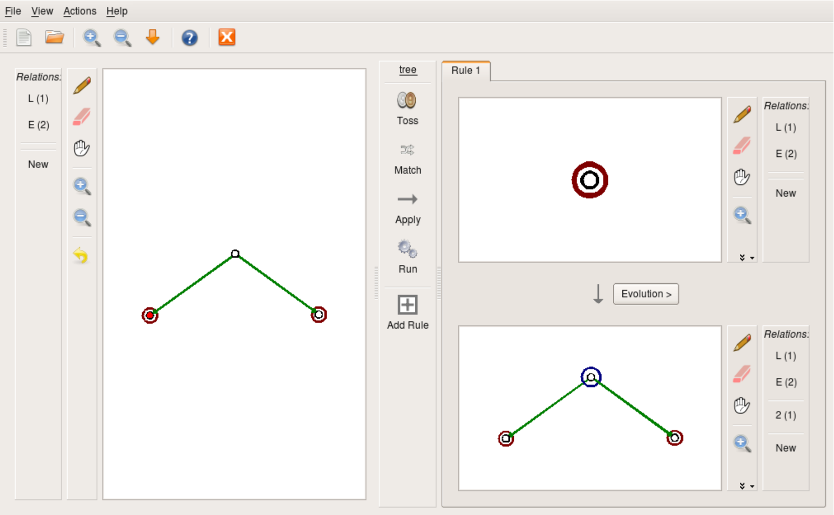


Structure Rewriting: Binary Tree



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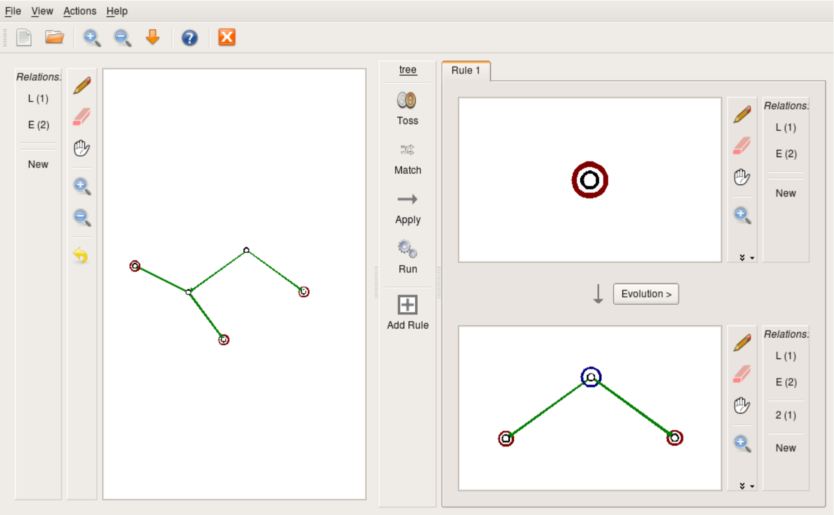

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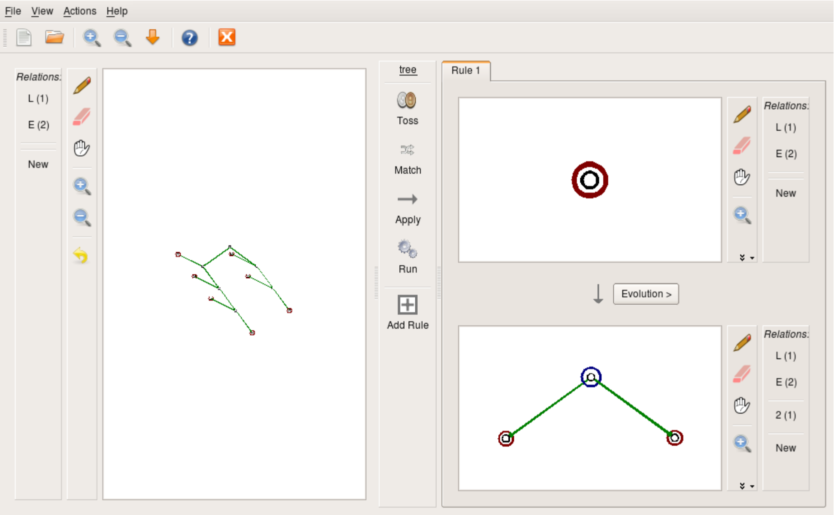
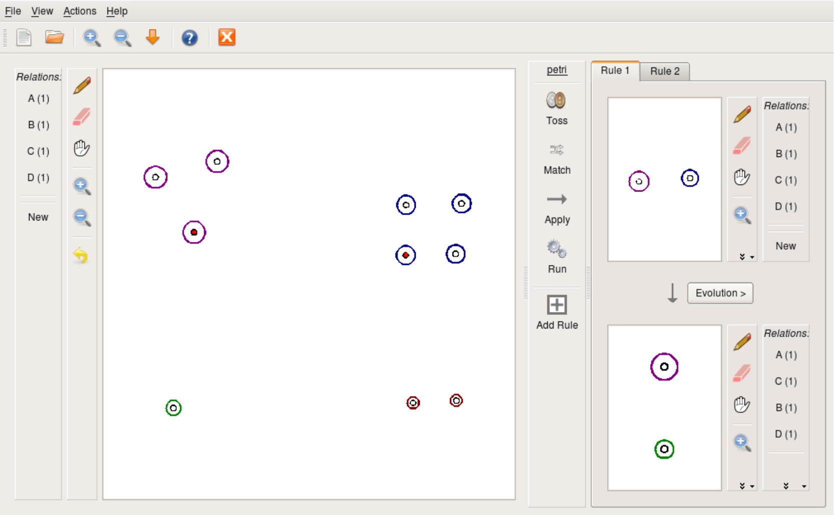


Structure Rewriting: Petri Net



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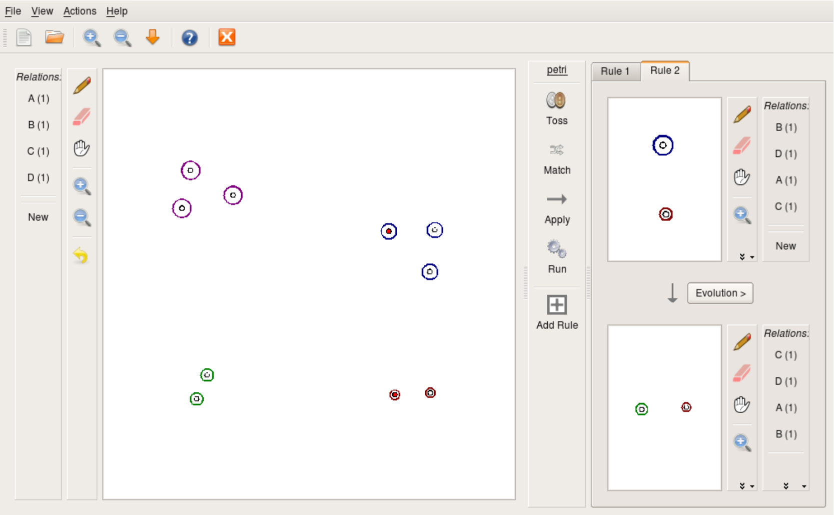

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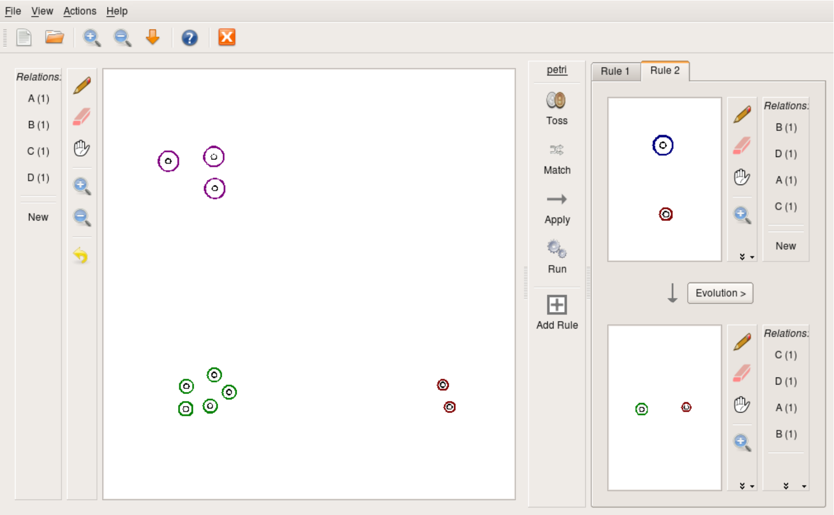
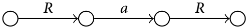
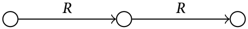


Simple Structure Rewriting

Separated Structures: no element is in two non-terminal relations **(Courcelle, Engelfriet, Rozenberg, 1991)**

Separated: 

Not Separated: 

Simple Rule $\mathfrak{L} \to \mathfrak{R}$: $\mathfrak{R}$ is **separated** and $\mathfrak{L}$ is a **single tuple in relation**

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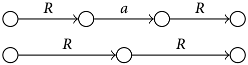

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Universal Rewriting and Limit Structures

- $\mathfrak{A}[\mathfrak{L} \to \mathfrak{R}]$ is $\mathfrak{A}$ with **all** occurrences of $\mathfrak{L}$ rewritten to $\mathfrak{R}$
- **Limit** of $\mathfrak{A}_0 \to \mathfrak{A}_1 \to \mathfrak{A}_2 \to \dots$: $(\bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} A_i, \bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} R_i)$

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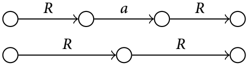

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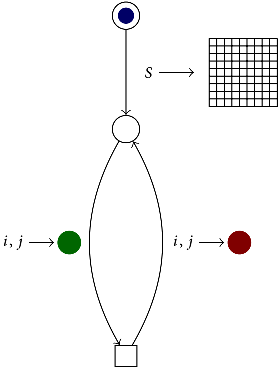
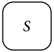
Why Universal Rewriting for Games?

- **In contrast** to **graph grammars** (single player)
- Establishing the winner if players **pick** embeddings is **undecidable:**
  - simulate **active context-free games** (thanks to **Anca Muscholl**)
- Choosing embedding can be allowed for **bounded number of non-terminals**

Example Game Played with Structures (Gomoku)





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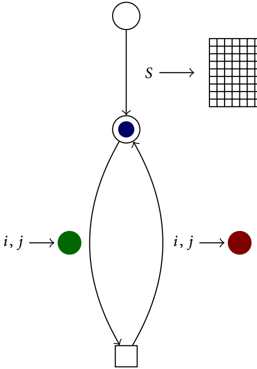

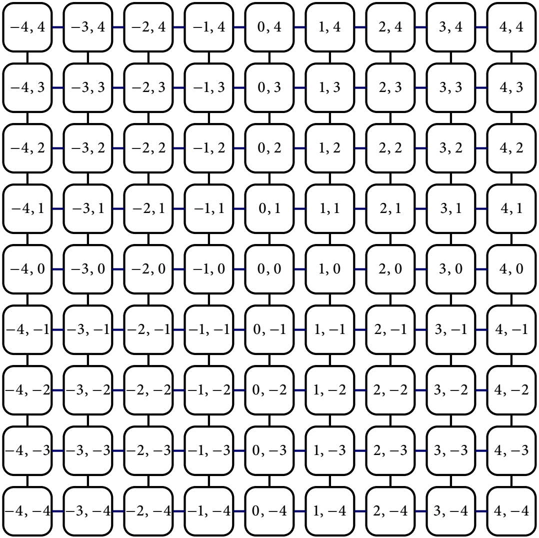

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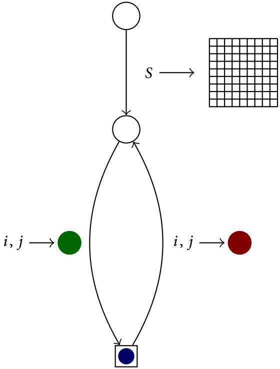

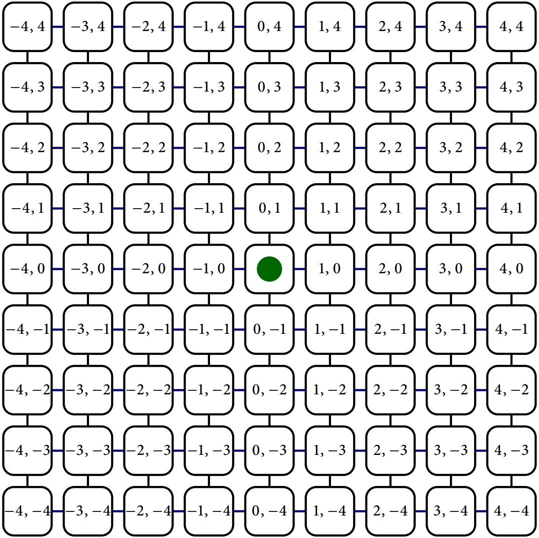

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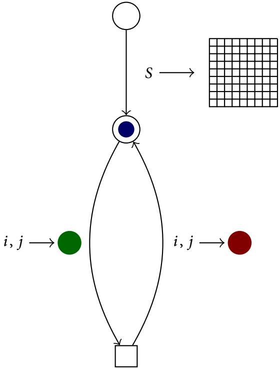

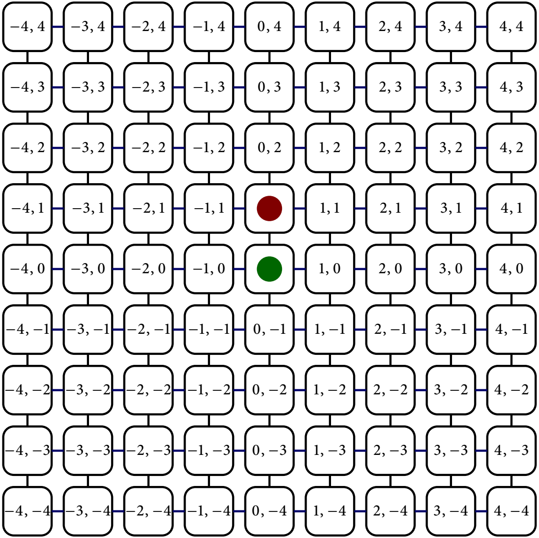

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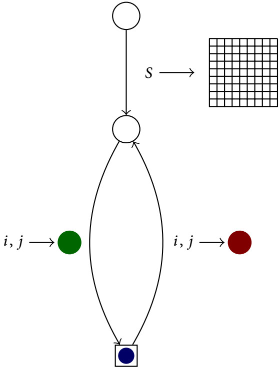

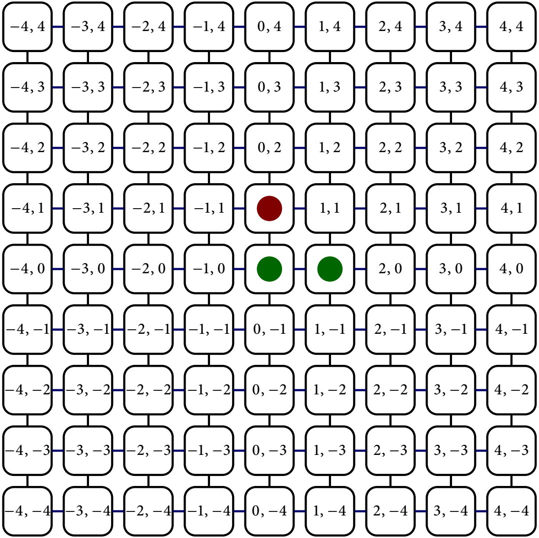

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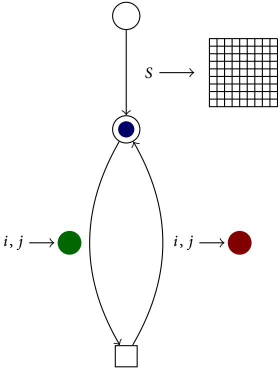

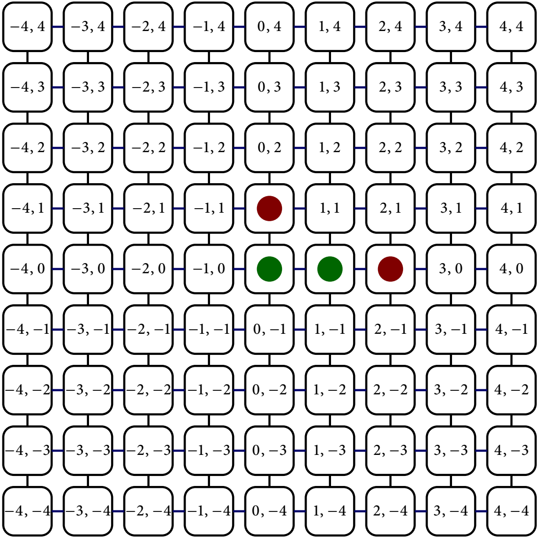

Main Result

Logics

- Properties of structures (states) expressed in MSO
- Temporal properties expressed in **the modal μ-calculus, L_μ**
- **Alternatively:** property of the limit structure expressed in MSO

Theorem

- *Let R be a **finite** set of **simple separated structure rewriting rules***
- *and φ be an $\mathsf{L}_\mu[\mathsf{MSO}]$ (or MSO) formula giving the **winning condition***

*Then the set $\{\pi \in R^\omega : (\lim) S(\pi) \models \varphi\}$ is **ω-regular**.*

Corollary

Establishing the winner of finite separated rewriting games is decidable.

Proof: How to Build a Graph

Pieces to build a graph:

- Bags of **single nodes** with **different colours** $1 \dots K$
- Paint to **change colour** of **all** nodes from i to j
- Edges to **connect** **all** nodes of colour i to **all** of colour j

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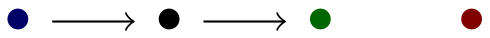

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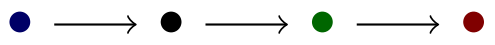

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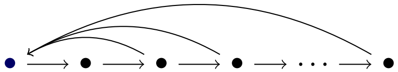

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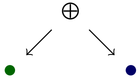

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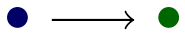

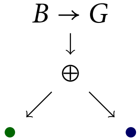

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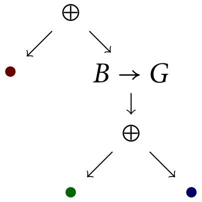

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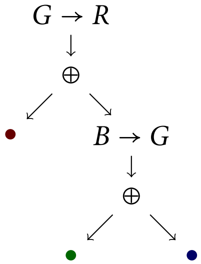

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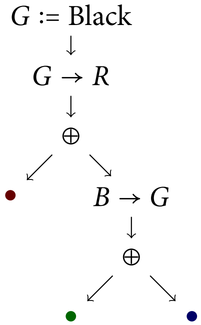

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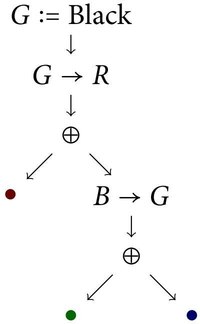

Theorem:
For every K there is an MSO-to-MSO **interpretation** $\mathcal{I}$ such that for all graphs $\mathcal{G}$ of **clique-width** $\leq K$ holds

$$\mathcal{I}(\mathcal{T}(\mathcal{G})) \cong \mathcal{G}$$

Proof: Separated Rewriting as a Tree

S

$\downarrow$

S

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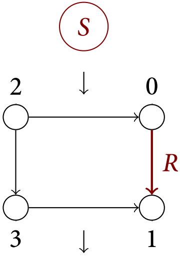

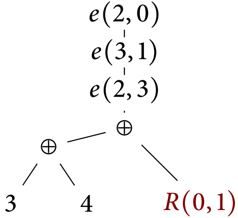

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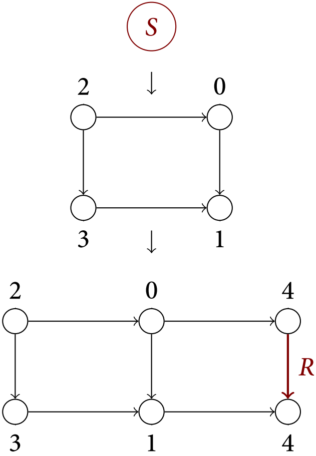
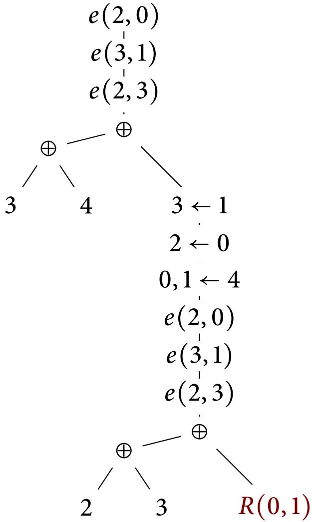
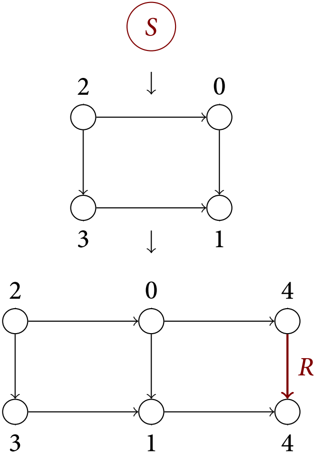
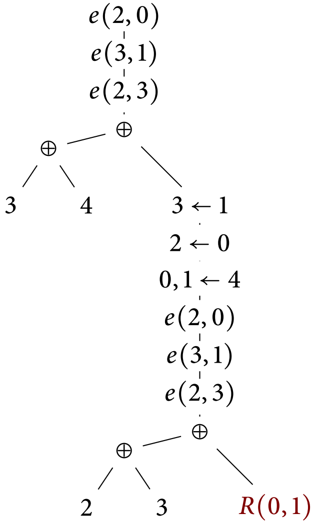




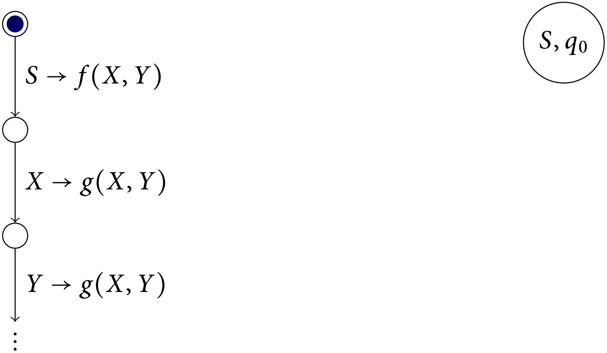
Proof: Separated Rewriting as a Tree



MSO-to-MSO **interpretation:** $\varphi \to \psi$



Proof: From Tree to Alternating Word Automata



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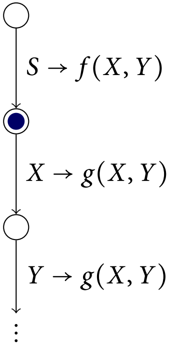

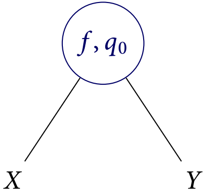

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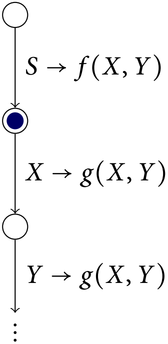

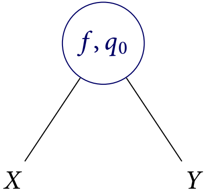

existential: pick transition

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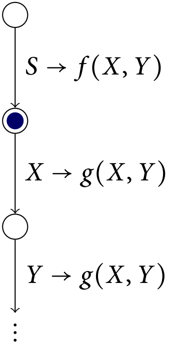

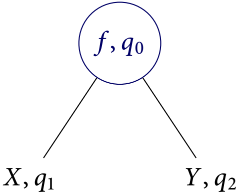

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$f, q_0 \to (q_1, q_2)$

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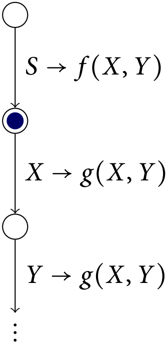

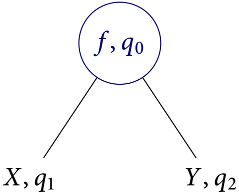

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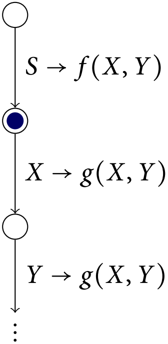

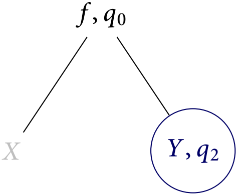

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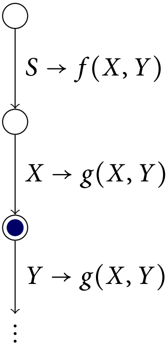

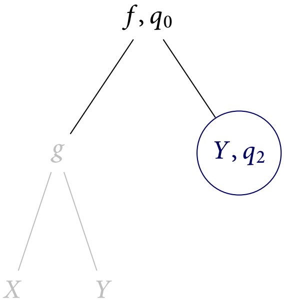

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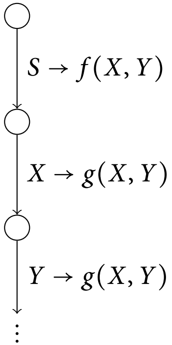

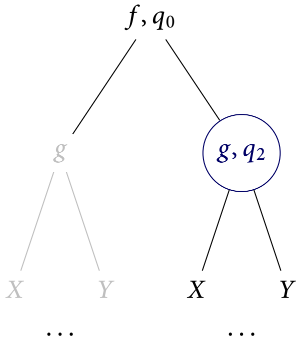

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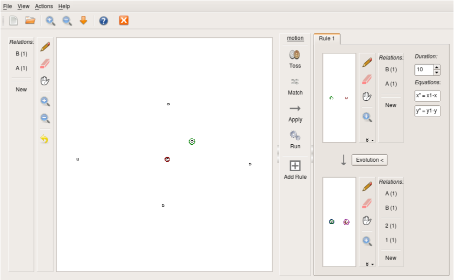

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Thank You