Playing Games when States have Rich Structure

Łukasz Kaiser

CNRS & LIAFA
Paris

GT Jeux Meeting

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Motivation

AlgoSyn: Algorithmic Synthesis of Reactive and Discrete-Continuous Systems

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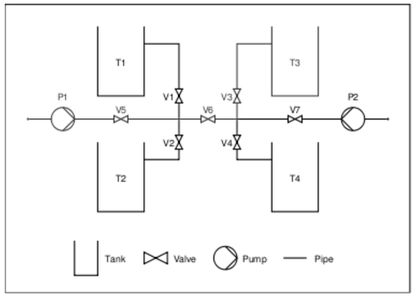
Part of a pumping station



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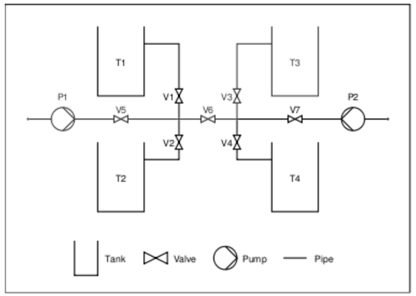
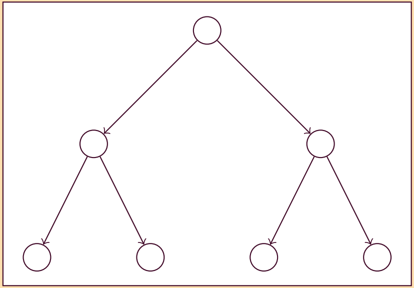


Structures we usually consider



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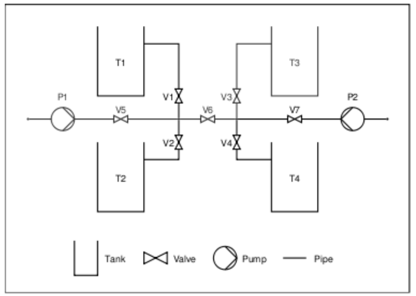

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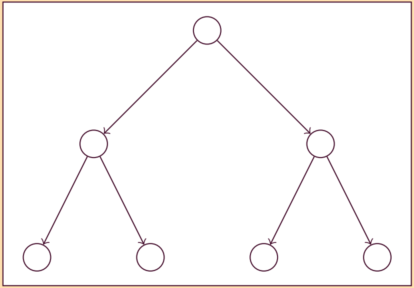


Can we model states by arbitrary relational structures?

(1) change described using appropriate rewriting rules

(2) properties given in MSO on structures + temporal logic for change

Overview

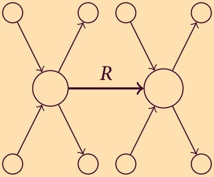
Structure Rewriting

Separated Games

Outlook

Structure Rewriting Rules

Rewriting Example



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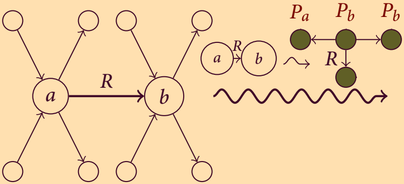

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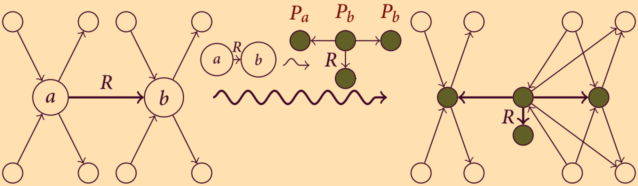

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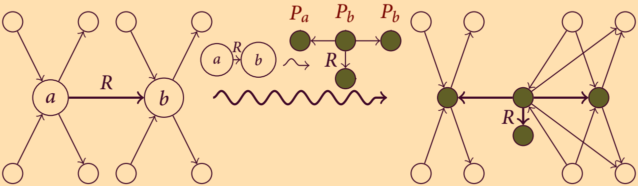

Relational Structures and Embeddings

$$\sigma \;:\quad \mathfrak{A} = (A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \dots, R_k^{\mathfrak{A}}) \quad\rightarrow\quad (B, R_1^{\mathfrak{B}}, R_2^{\mathfrak{B}}, \dots, R_k^{\mathfrak{B}}) = \mathfrak{B}$$

Embedding: σ is **injective** and $R_i^{\mathfrak{A}}(a_1, \dots, a_{r_i}) \Leftrightarrow R_i^{\mathfrak{B}}(\sigma(a_1), \dots, \sigma(a_{r_i}))$

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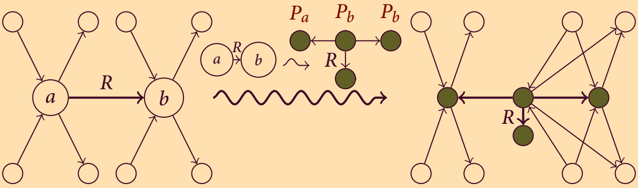

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Rewriting Definition

$\mathfrak{B} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$ iff $B = (A \setminus \sigma(L)) \dot\cup R$ and,
for $M = \{(r, a) \mid a = \sigma(l), r \in P_l^{\mathfrak{R}}$ for some $l \in L\} \cup \{(a, a) \mid a \in A\}$,

$$(b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{B}} \ \Leftrightarrow \ (b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{R}} \textbf{ or } (b_1 M \times \ldots \times b_{r_i} M) \cap R_i^{\mathfrak{A}} \neq \emptyset.$$

(in the second case at least one $b_j \notin \mathfrak{A}$)

Structure Rewriting Games

Game arena (of a two-player zero-sum game) is a **directed graph** with:

- vertices partitioned into positions of **Player 0** and **Player 1**
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- **Existential**: $\mathfrak{A}_{\text{next}} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$, the player chooses the embedding σ
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Winning conditions:

- L_μ (or temporal) formula ψ with **MSO sentences** for predicates, or
- MSO formula φ to be evaluated on the **limit** of the play
  Limit of $\mathfrak{A}_0\mathfrak{A}_1\mathfrak{A}_2\ldots = \left(\bigcup_{n\in\mathbb{N}}\bigcap_{i\geq n} A_i,\ \bigcup_{n\in\mathbb{N}}\bigcap_{i\geq n} R^{\mathfrak{A}_i}\right)$
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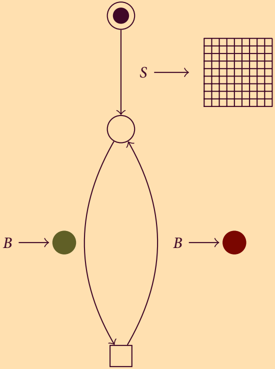
Motivation: many questions are **naturally defined as such games**:
constraint satisfaction, model checking, graph measures, games people play

Example Game: Gomoku (Connect–5)



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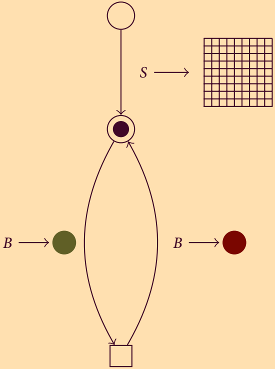

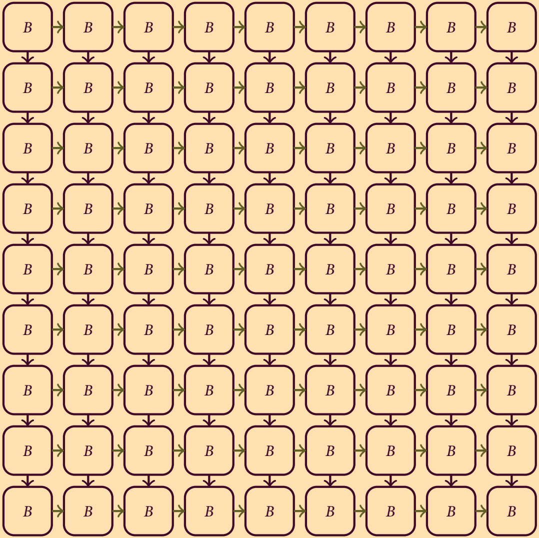

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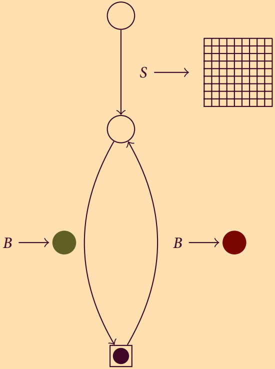

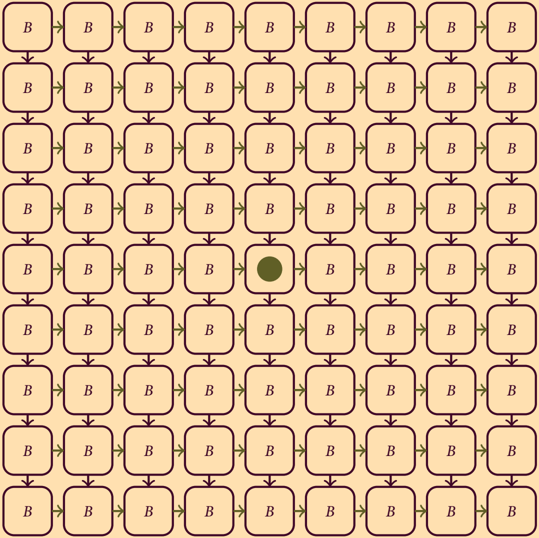

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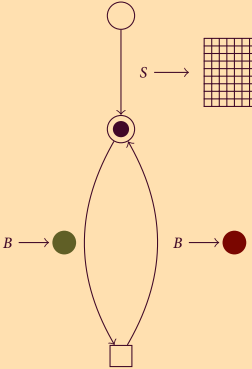

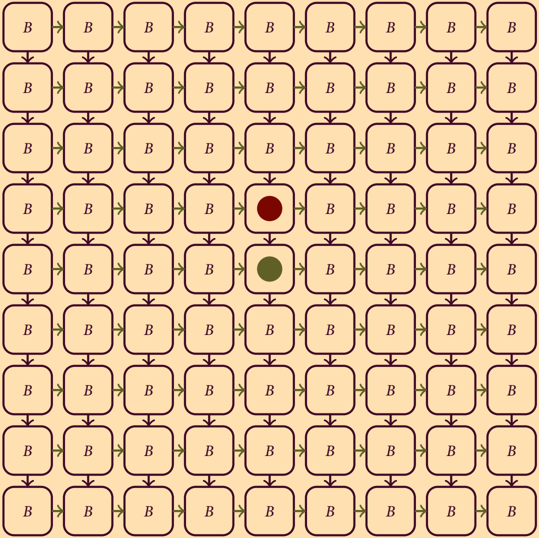

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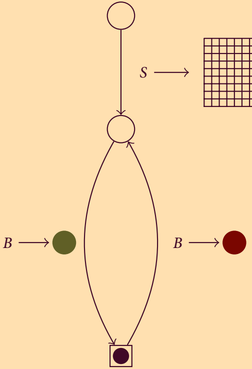

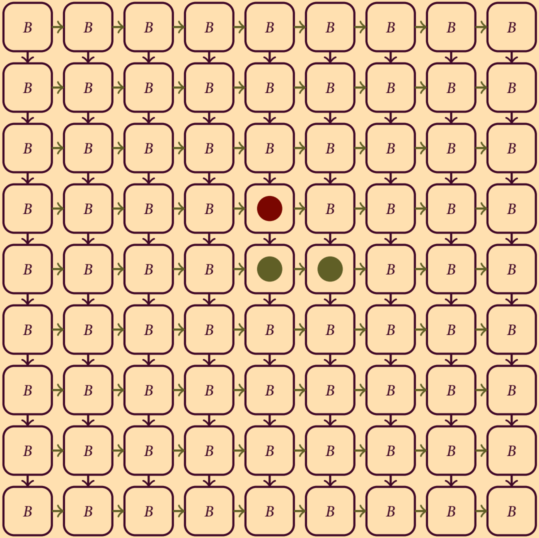

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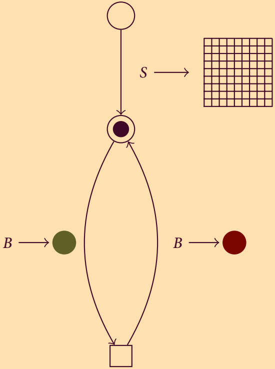

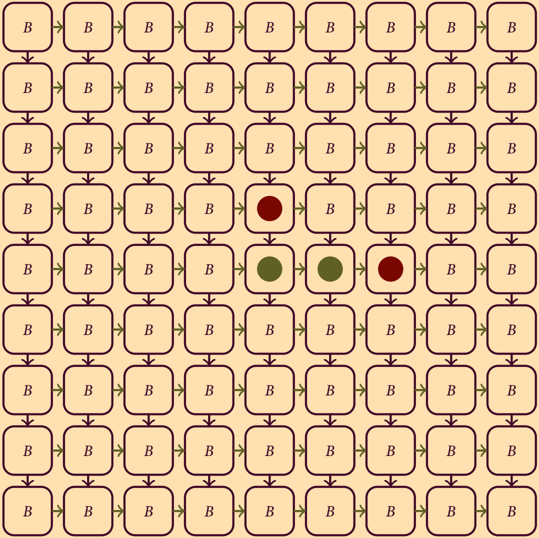

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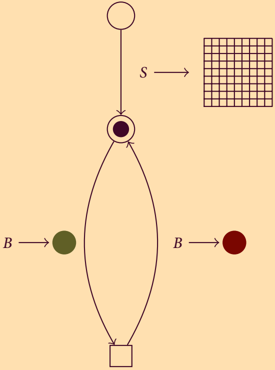

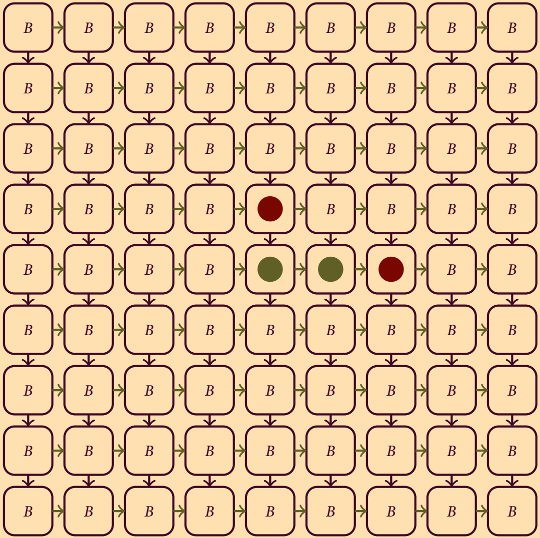

$$\exists x_1 \dots x_5 \Big(\bigwedge_{1\le i\le 5} G(x_i) \wedge \Big(\bigwedge_{1\le i\le 5} R(x_i, x_{i+1}) \vee \bigwedge_{1\le i\le 5} C(x_i, x_{i+1}) \vee \bigwedge_{1\le i\le 5} \exists y(R(x_i, y) \wedge C(y, x_{i+1})) \vee \bigwedge_{1\le i\le 5} \exists y(R(x_i, y) \wedge C(x_{i+1}, y))\Big)\Big)$$

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Simple Structure Rewriting

Separated Structures: no element is in two **non-terminal** relations
(Courcelle, Engelfriet, Rozenberg, 1991)

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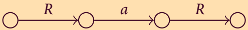

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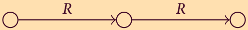

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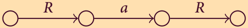

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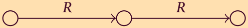

Simple Rule $\mathfrak{L} \to \mathfrak{R}$: $\mathfrak{R}$ is **separated** and $\mathfrak{L}$ is a **single tuple in relation**

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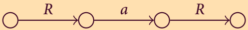
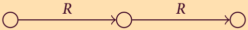
Separated: 

Not Separated: 

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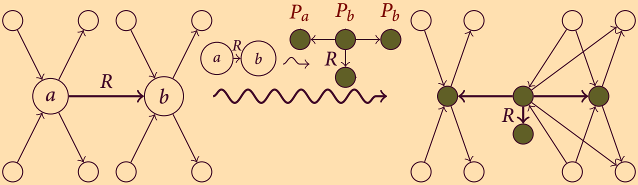

Decidability of Simple Rewriting Games

Logics

- $L_\mu[\mathrm{MSO}]$**:** Temporal properties expressed in L_μ (subsumes LTL) with properties of structures (states) expressed in MSO
- $\lim \mathrm{MSO}$**:** Property of the limit structure expressed in MSO

Theorem

- *Let R be a **finite** set of (universal) simple structure rewriting rules,*
- *and φ be an $L_\mu[\mathrm{MSO}]$ or $\lim \mathrm{MSO}$ formula.*

*Then the set $\{\pi \in R^\omega \,:\, (\lim) S(\pi) \models \varphi\}$ is ω-**regular**.*

Corollary

Establishing the winner of (universal) finite simple rewriting games is decidable. The winner has a winning strategy of a simple form.

Proof: Interpreting a Structure in a Tree

Description of how to build $\mathfrak{A}$ is a tree $\mathcal{T}(\mathfrak{A})$ with:

- **Leafs** of **different colours** $1 \ldots k$
- $\oplus$ representing **disjoint sum**
- $i \leftarrow j$ to **change colour** of **all** nodes from i to j
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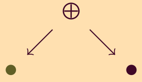

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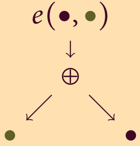

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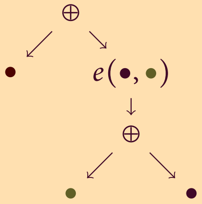

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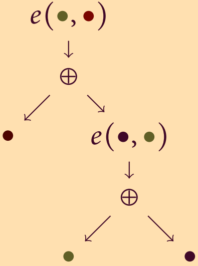

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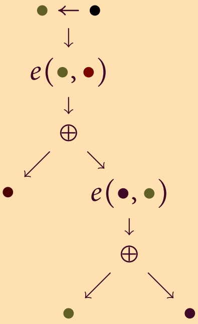

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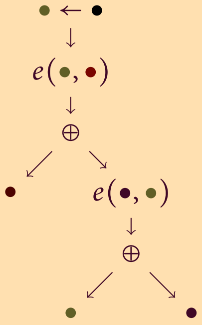

Theorem:
For every k there is an MSO-**to**-MSO **interpretation** $\mathcal{I}$ such that for all structures $\mathfrak{A}$ of **clique-width** $\leq k$ holds $\mathcal{I}(\mathcal{T}(\mathfrak{A})) \cong \mathfrak{A}$.

Proof: Simple Rewriting in the Tree

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$\downarrow$

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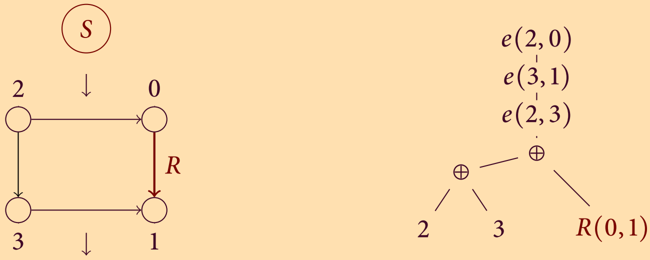

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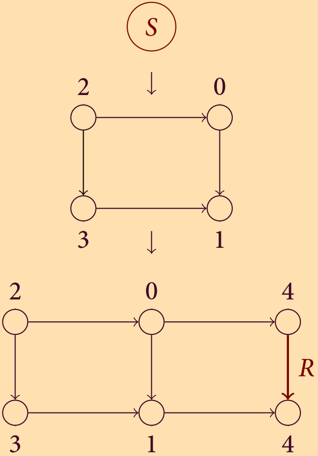

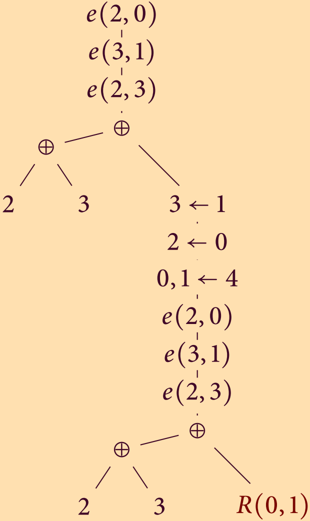

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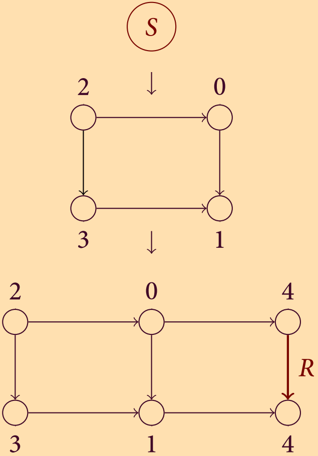
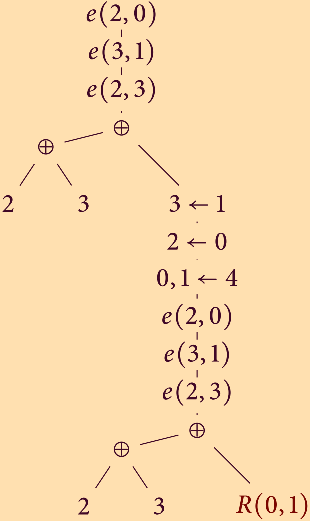
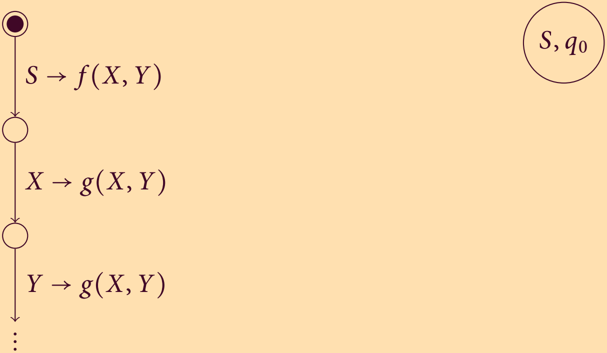




MSO-to-MSO interpretation: $\varphi \to \psi$

Proof: From Tree to Alternating Word Automata



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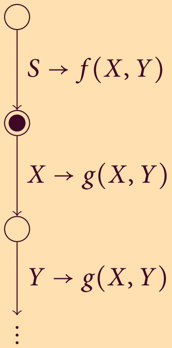

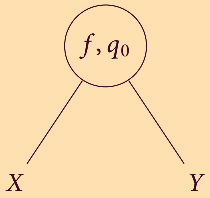

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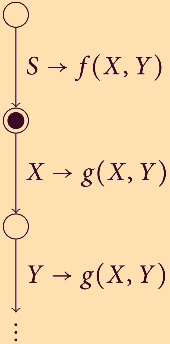

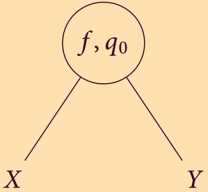

existential: pick transition

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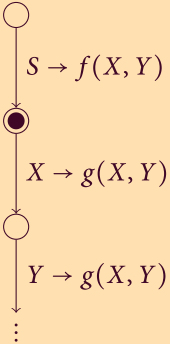

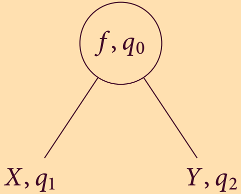

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$$f, q_0 \to (q_1, q_2)$$

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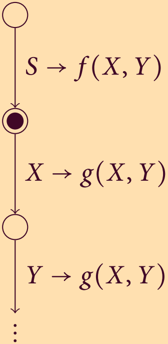

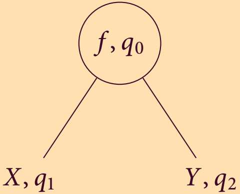

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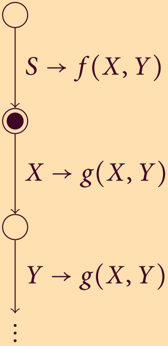

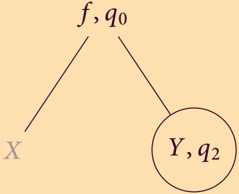

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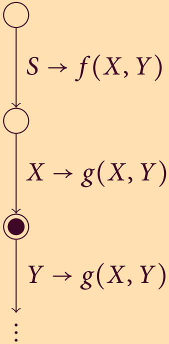

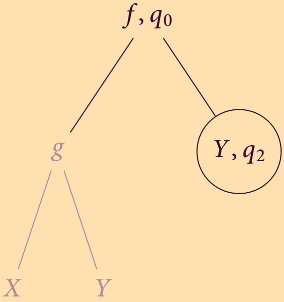

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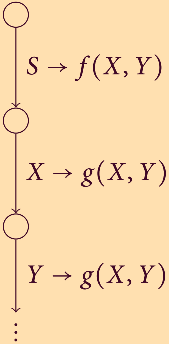

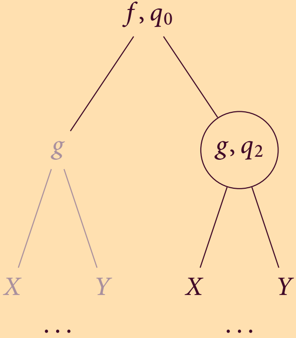

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Overview

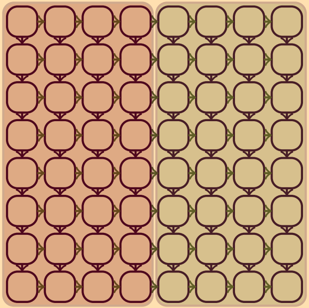
Structure Rewriting

Separated Games

Outlook

Imperfect Information

How to Represent Imperfect Information?



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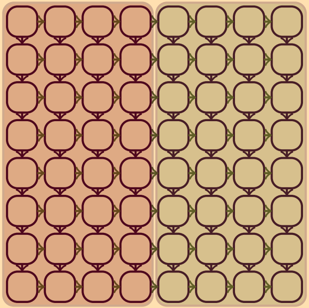

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Application: **abstraction** and abstraction refinement for **complex games**.

Strategies and Higher-Order Games

Strategies in Separated Games

- **Winning positions** can be defined in **MSO**
- What is a **simple form of strategy**?

  $\mathfrak{A}_1 \to \mathfrak{A}_2 \to \ldots \to \mathfrak{A}_n \to ?$
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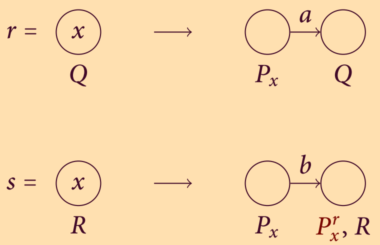
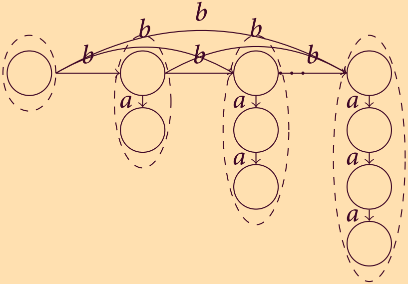
Higher-Order Structures and Rewriting





Does this correspond to **higher-order pushdown systems** and **strategies**?

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Structure Rewriting Games

- **General** model of games with **structured states**
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Thank You