Games Played with Hypergraphs

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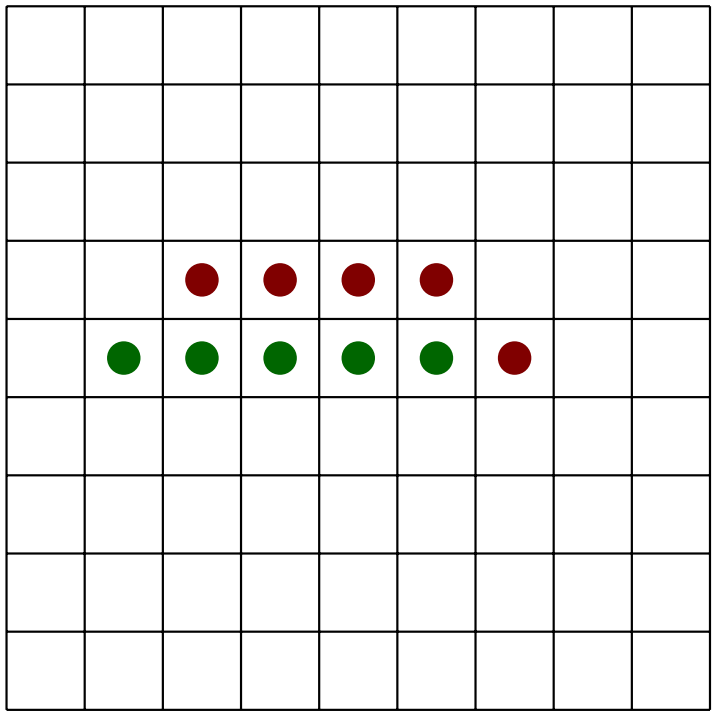
RWTH Aachen

Games 2008, Warsaw

How People Play Games

Gomoku (five in a row)



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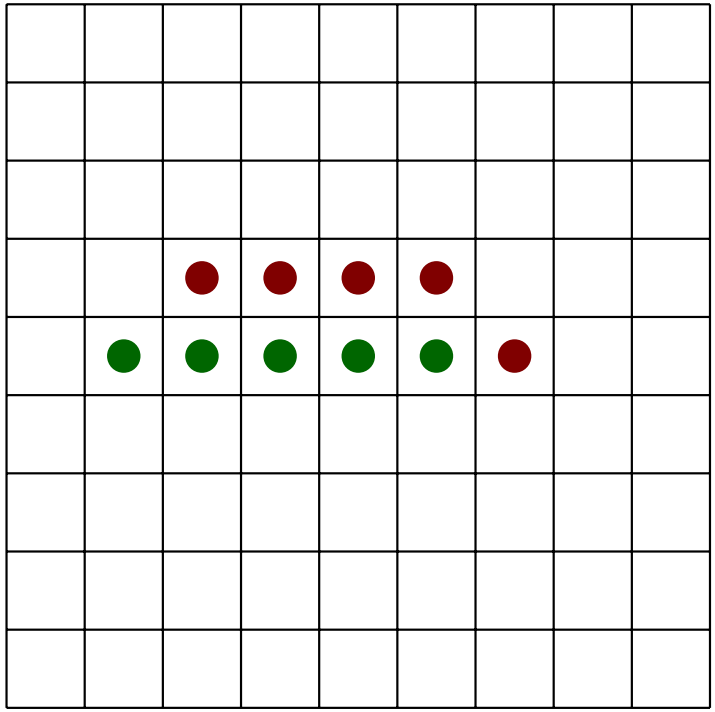
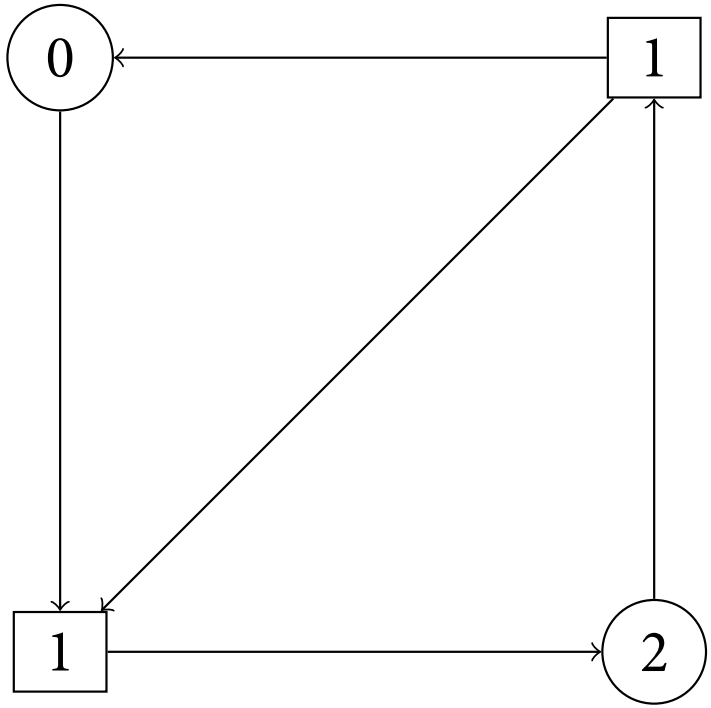


Parity game



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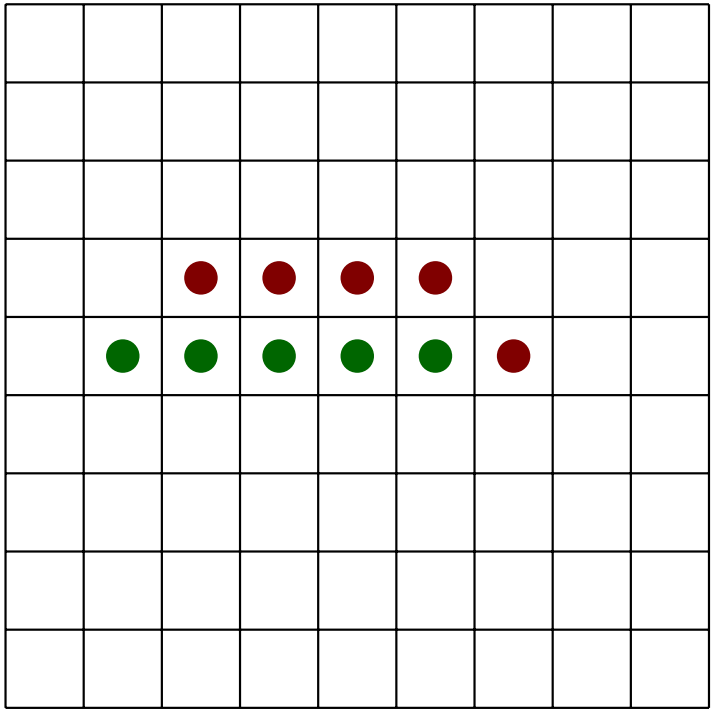

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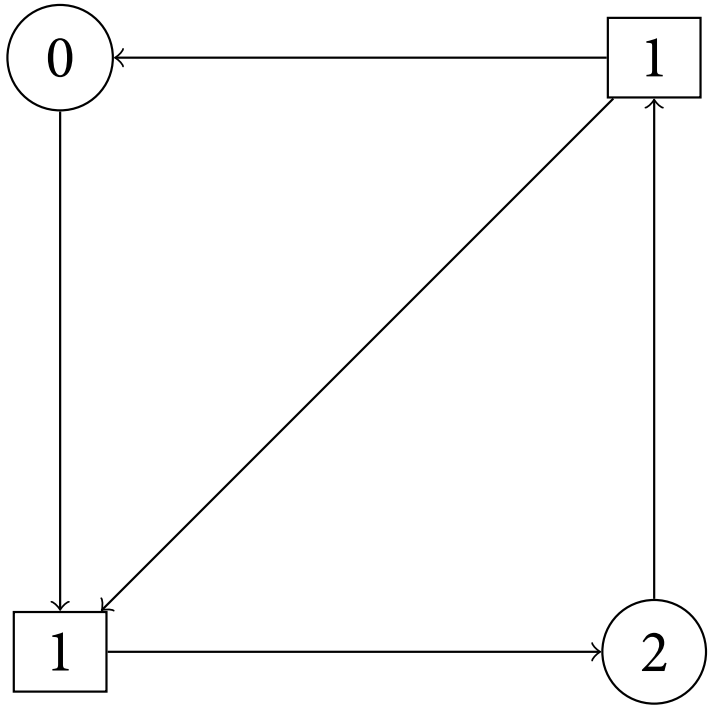


When state space is infinite, we often (again) consider its structure

- to construct **algorithms**
- to prove **interesting properties**

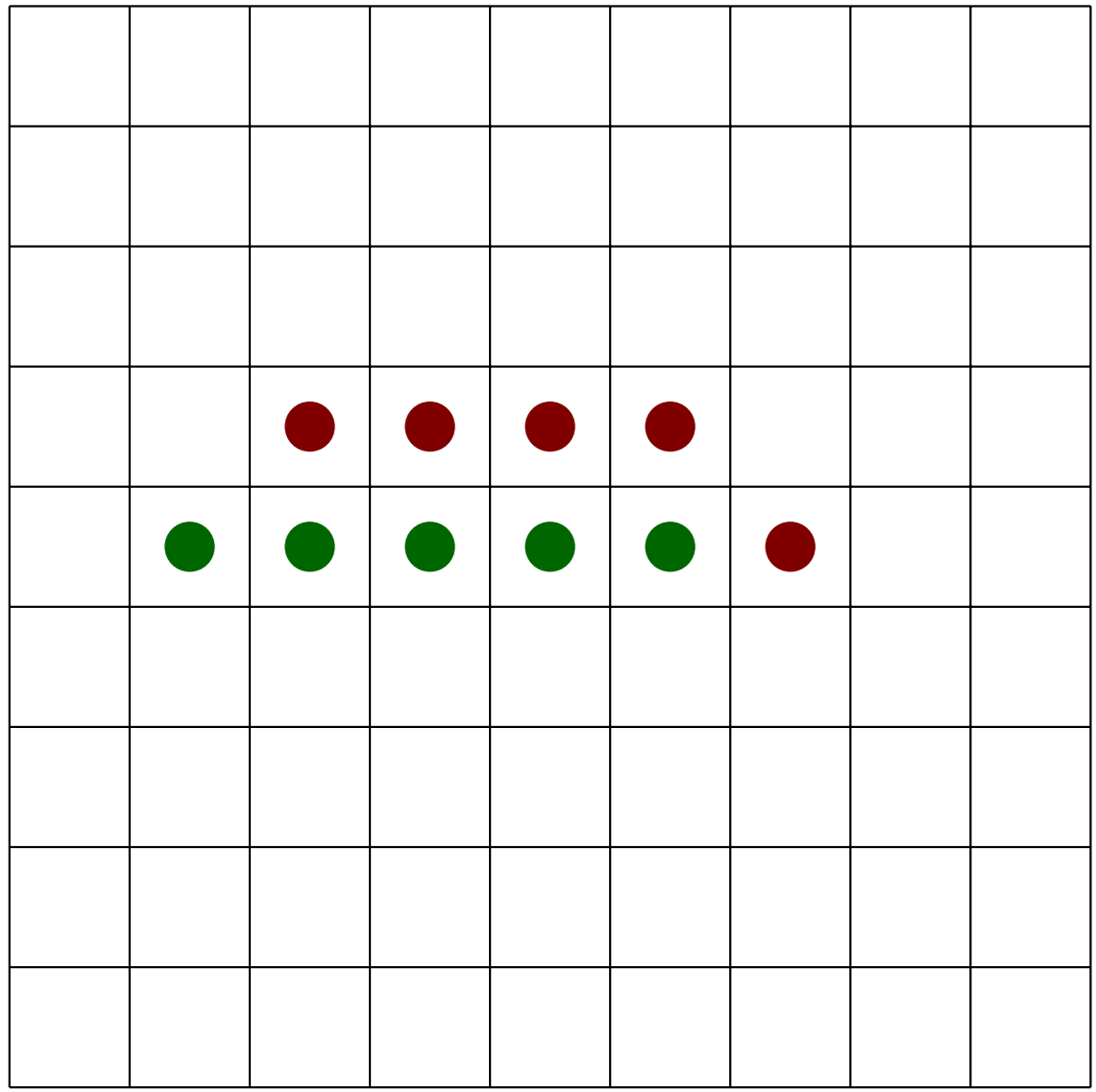
Example: pushdown games

A Natural Game Presentation

Gomoku Rules

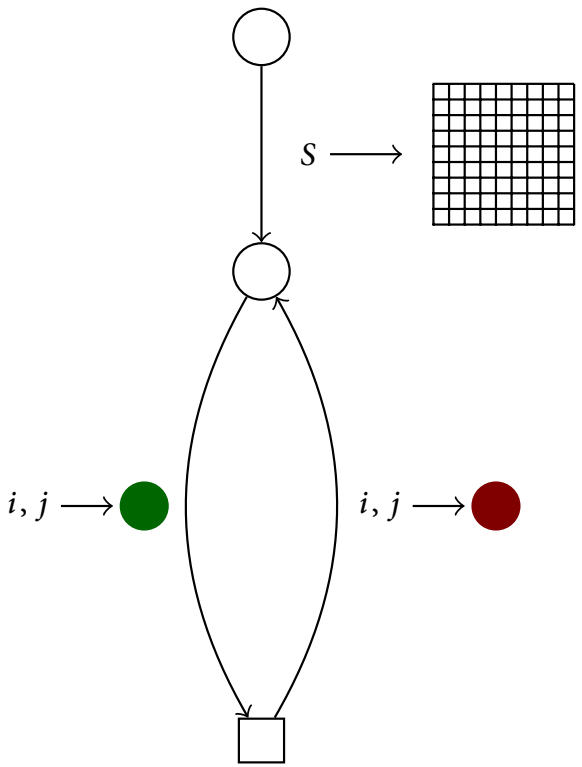
- 9×9 board
- Player 0 puts
- Player 1 puts
- Players alternate

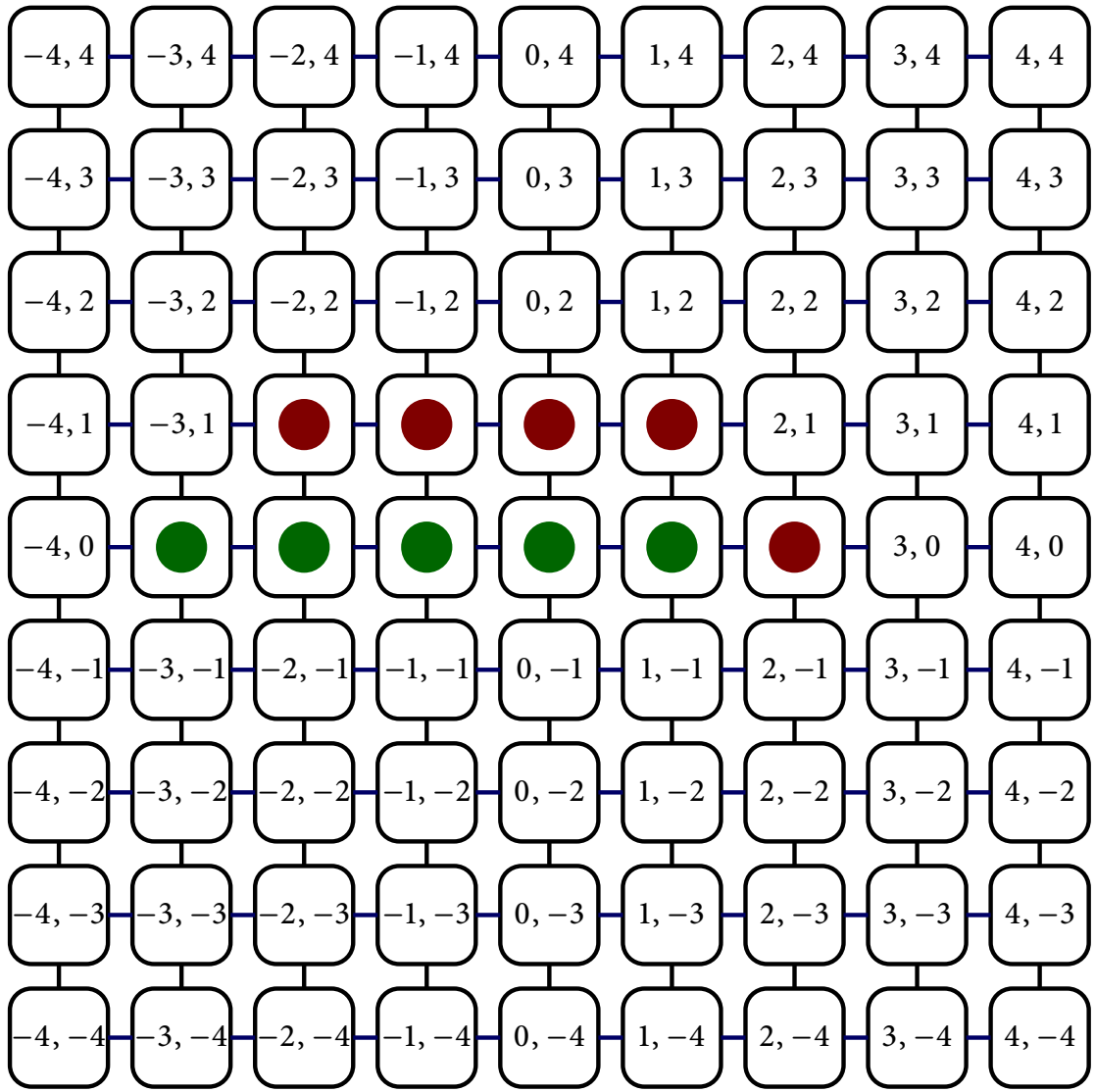
Gomoku Board



A Natural Game Presentation

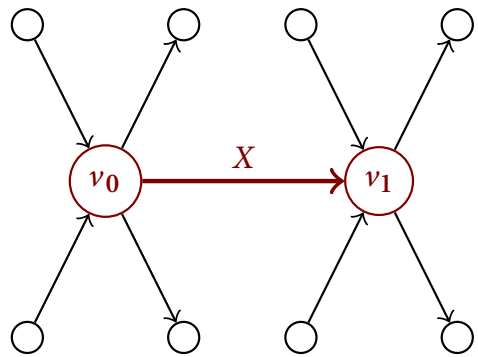
Representing the Rules and the Gomoku Board





Games Played with Hypergraphs

Handle Rewrite Rules (Courcelle, Engelfriet, Rozenberg, 1991)



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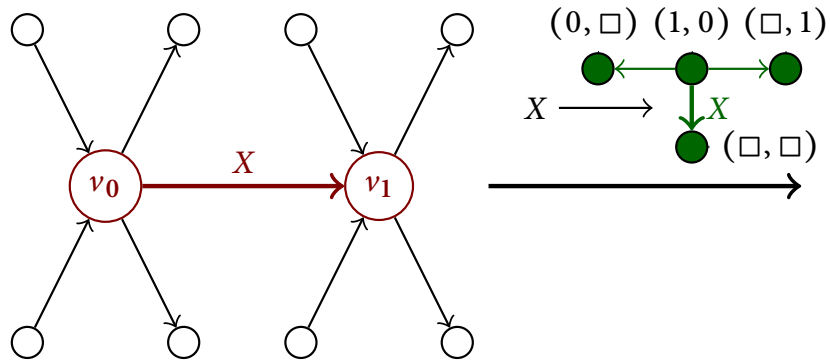

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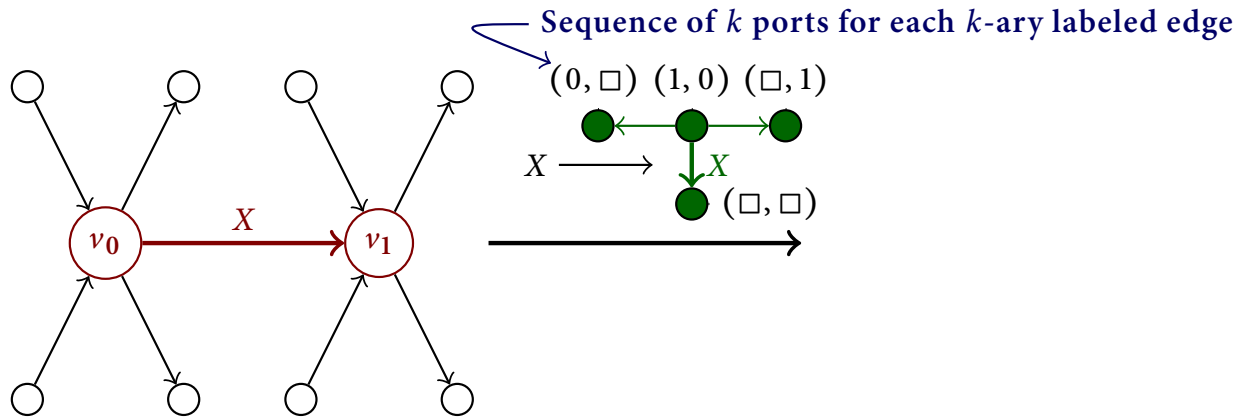

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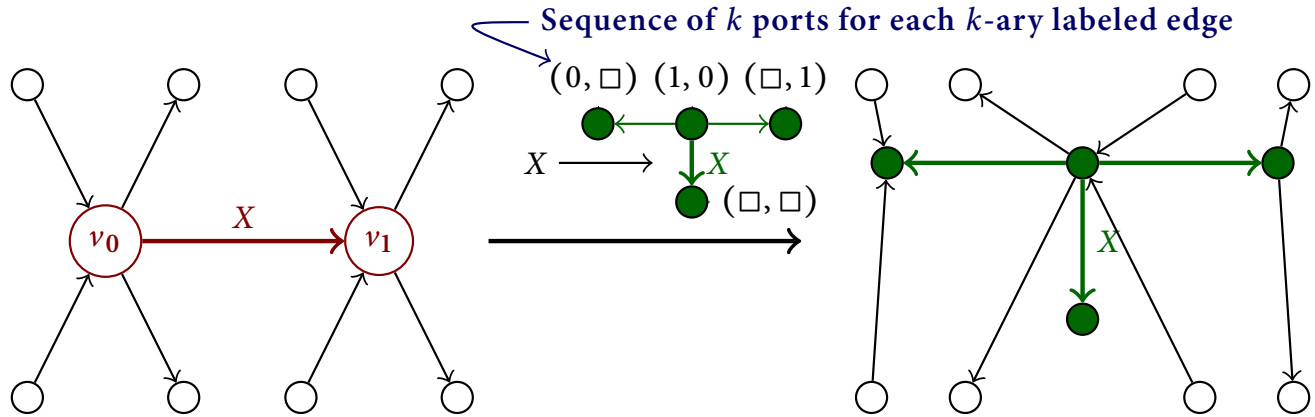

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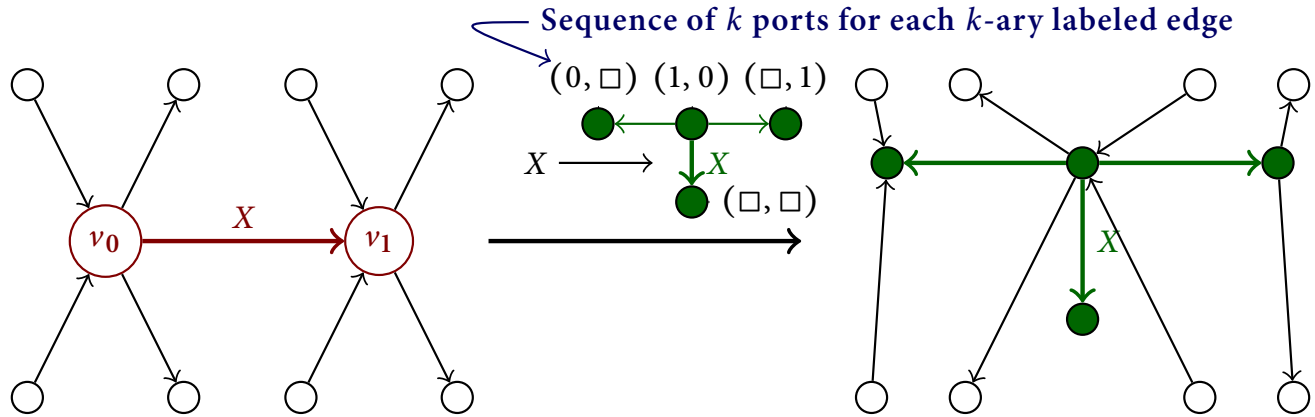
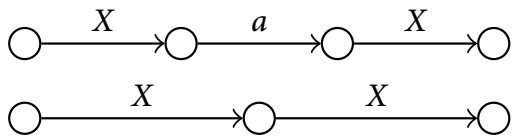


Separated Hypergraphs: no vertex is incident to two non-terminal edges

Separated:

Not Separated:



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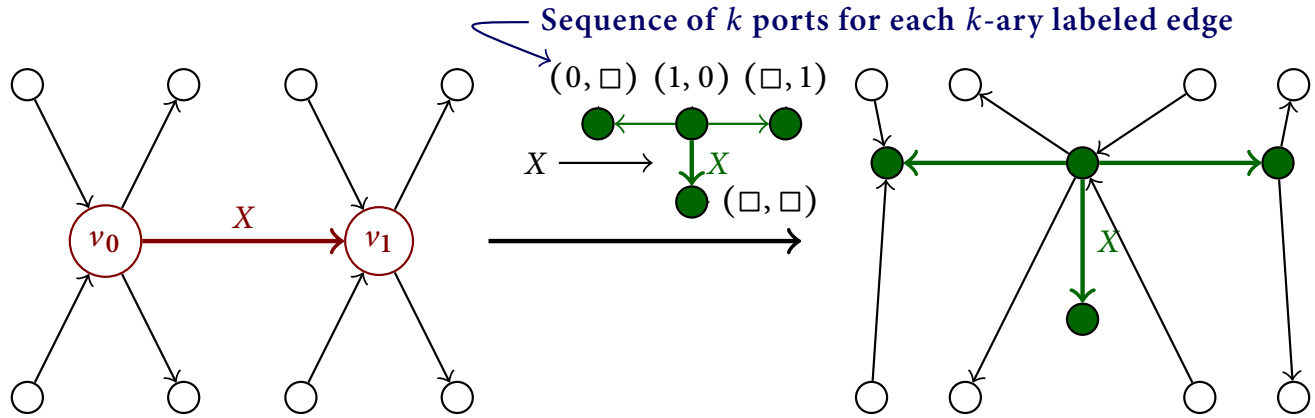

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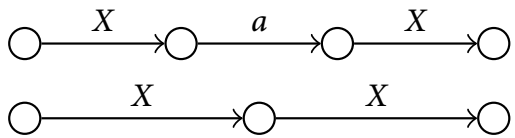


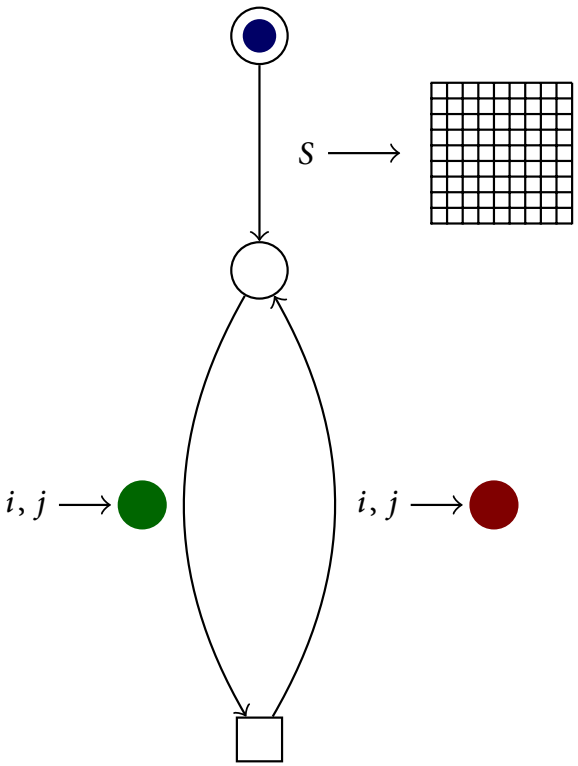
Rewriting Sequences and Limit Hypergraphs

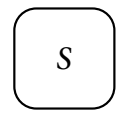
- $G[X \to H]$ is G with **all*** occurrences of X rewritten to H

  * if players **pick** positions: **undecidable**, see **Active context-free games**, thanks to **Anca Muscholl**

- **Limit** of $G_0 \to G_1 \to G_3 \to \ldots$: $(\bigcup_{n \in \mathbb{N}} \bigcap_{i \geq n} V_i, \bigcup_{n \in \mathbb{N}} \bigcap_{i \geq n} E_i)$

Example: Playing Gomoku





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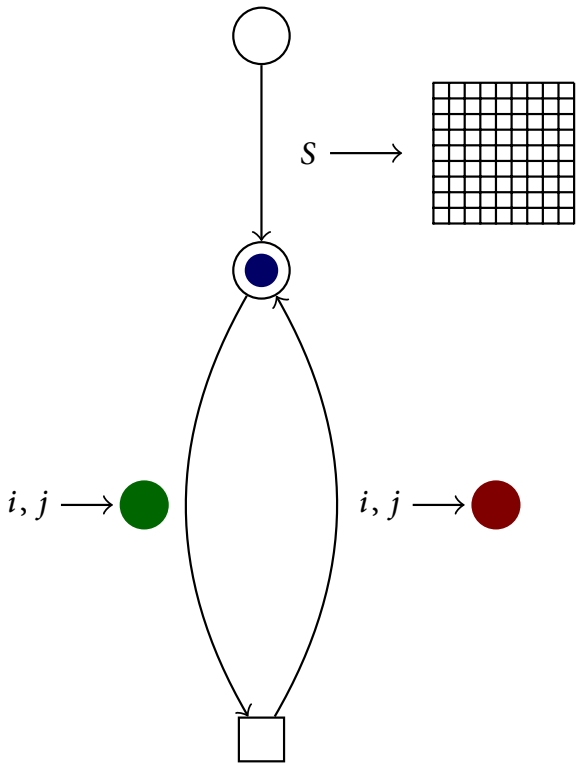

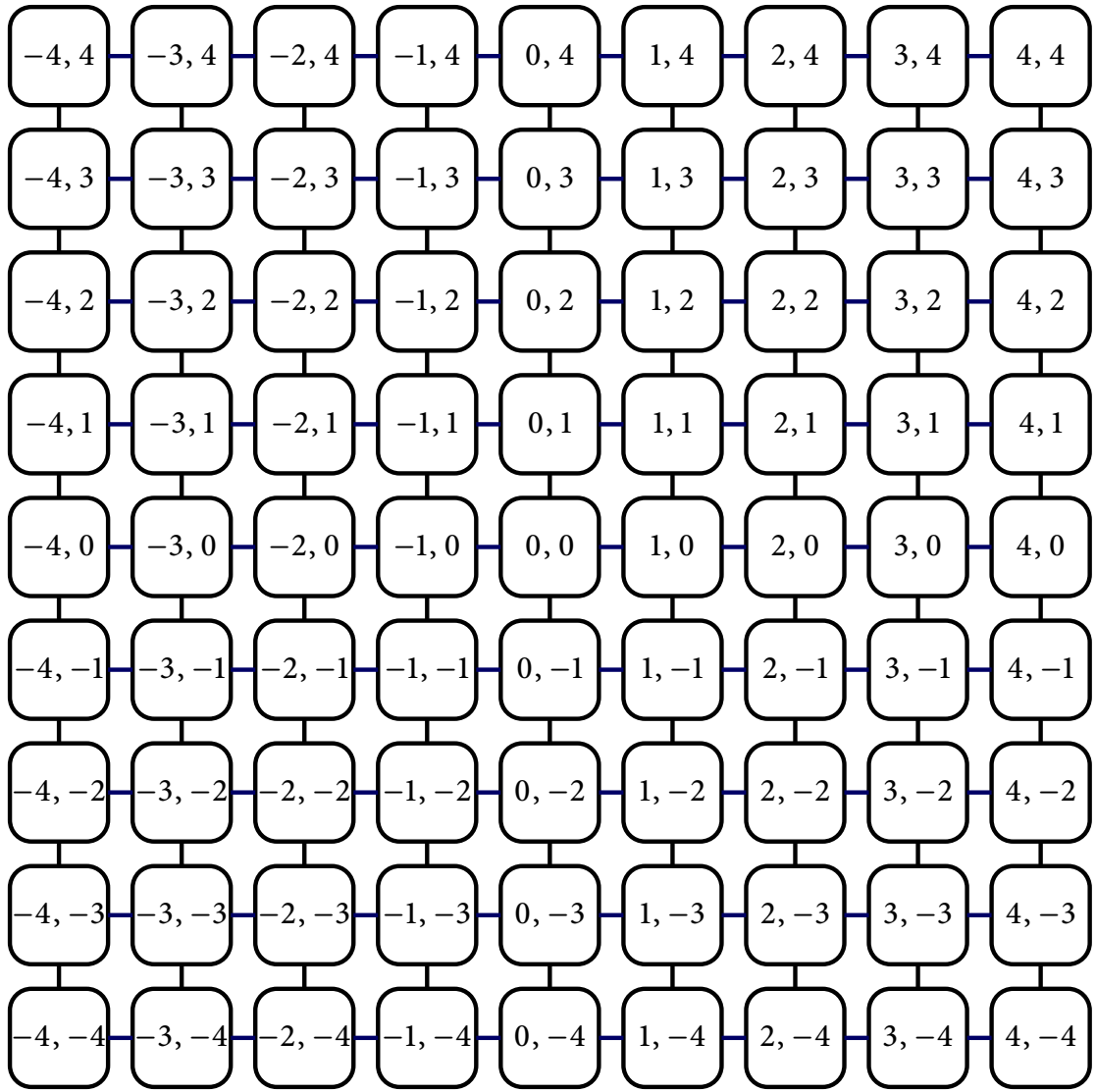

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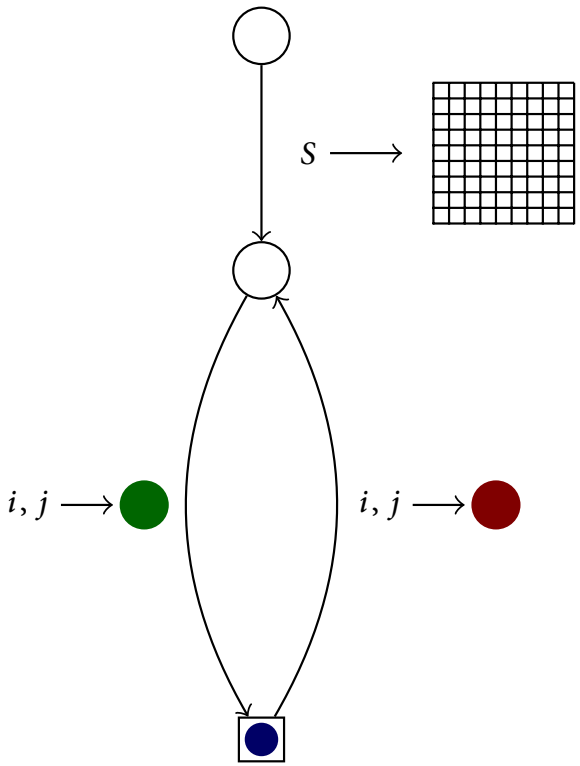

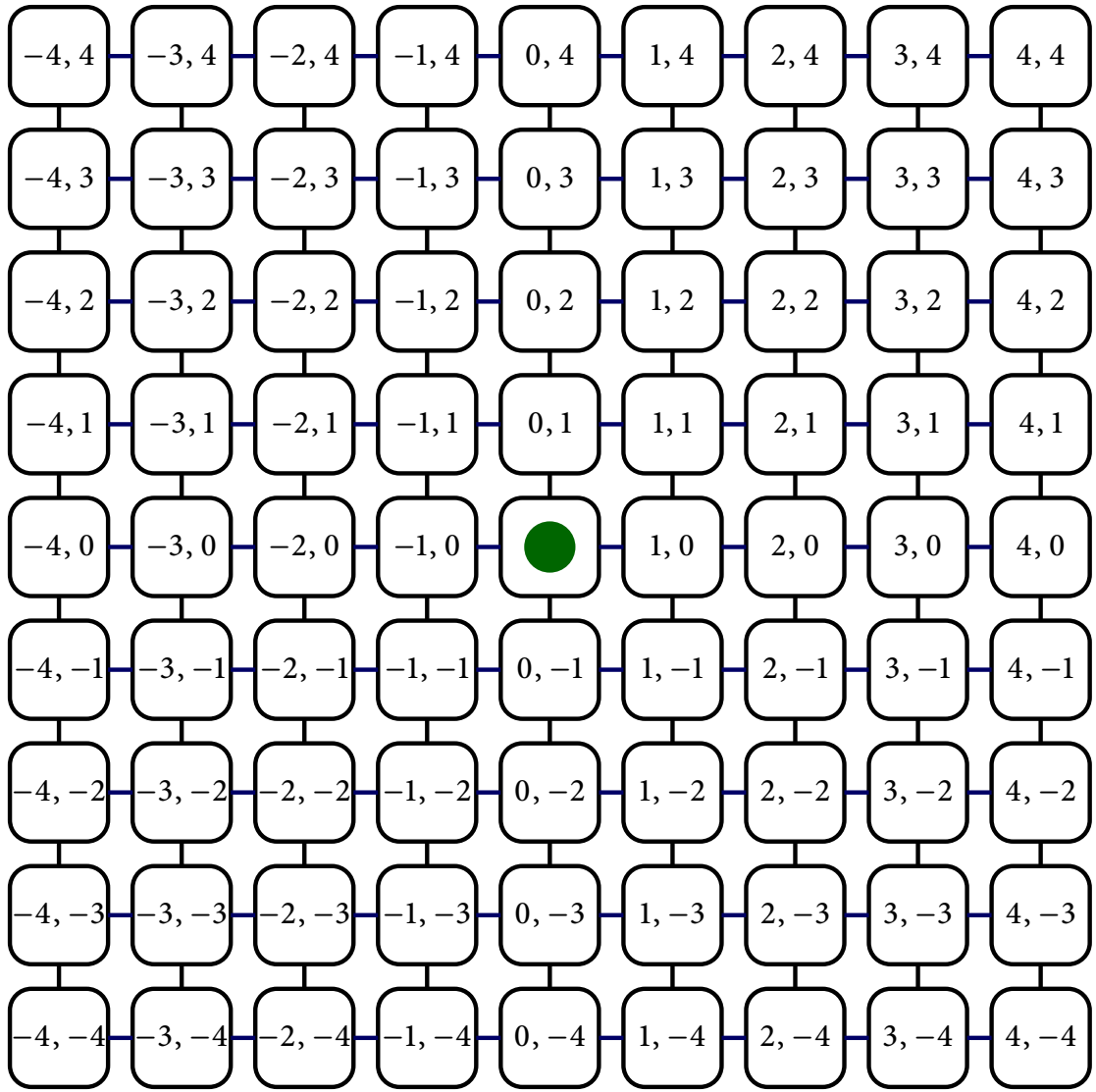

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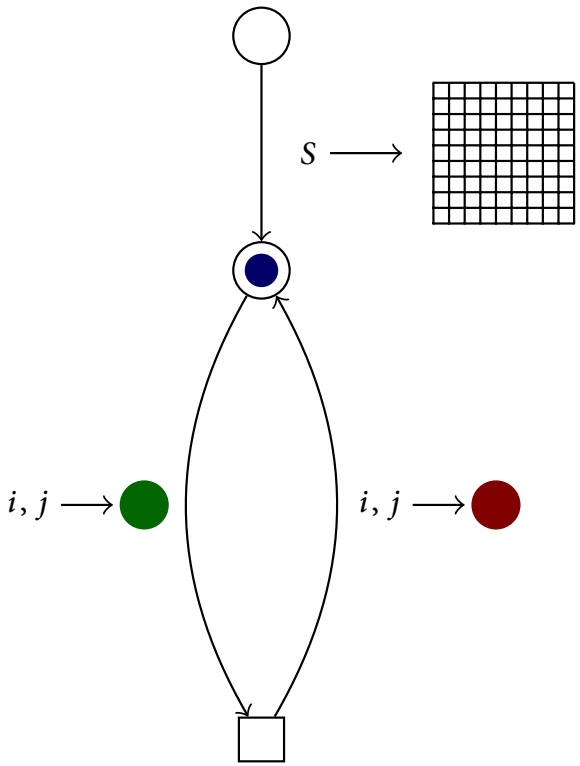

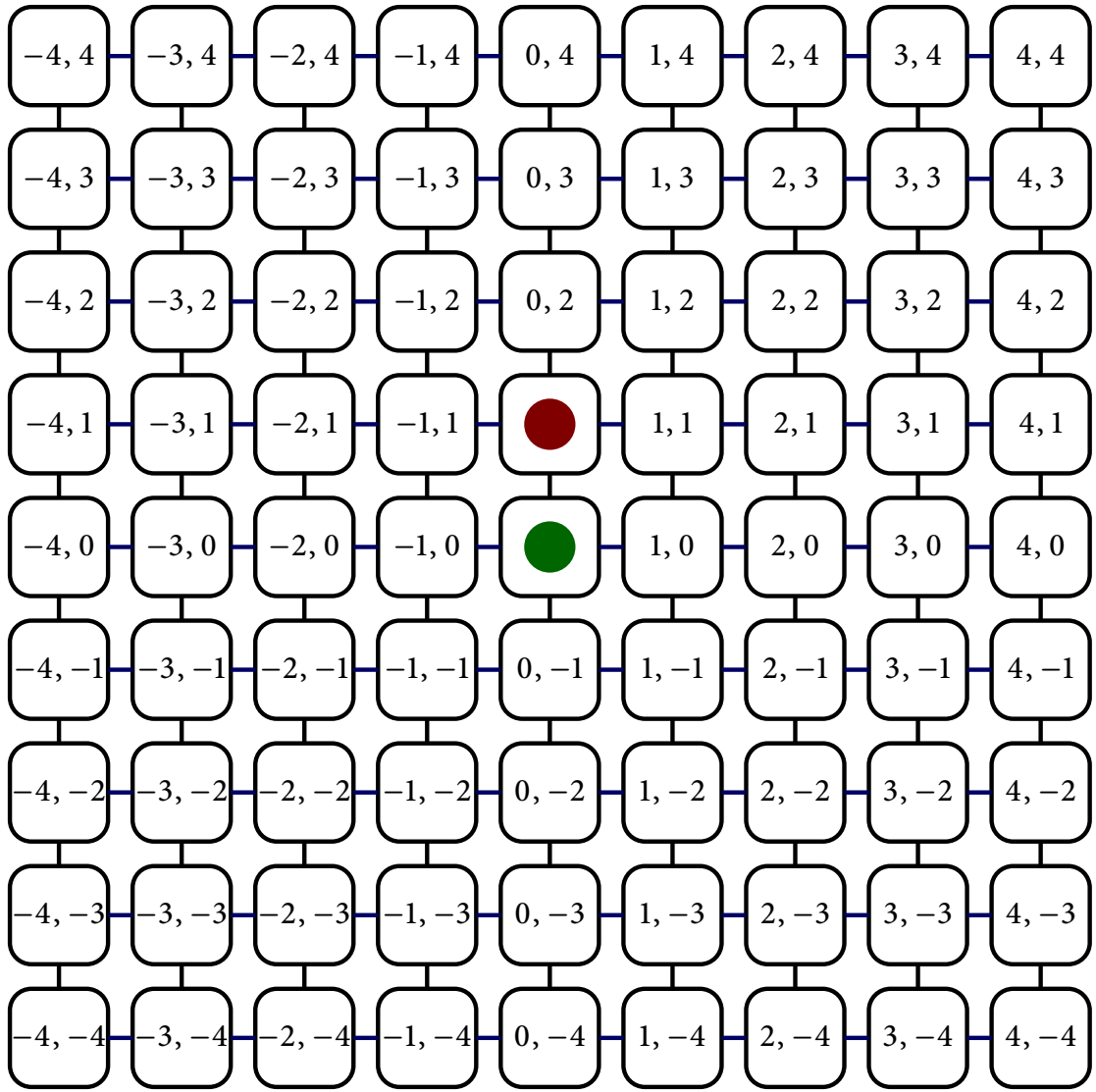

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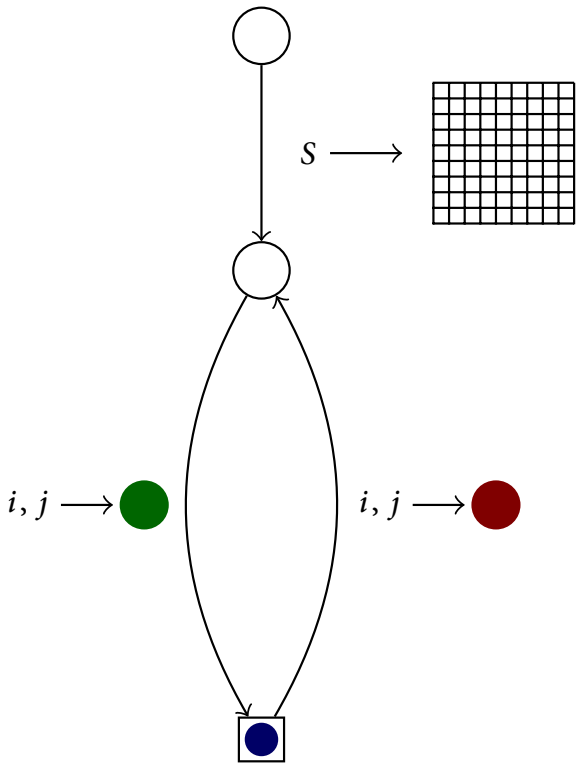

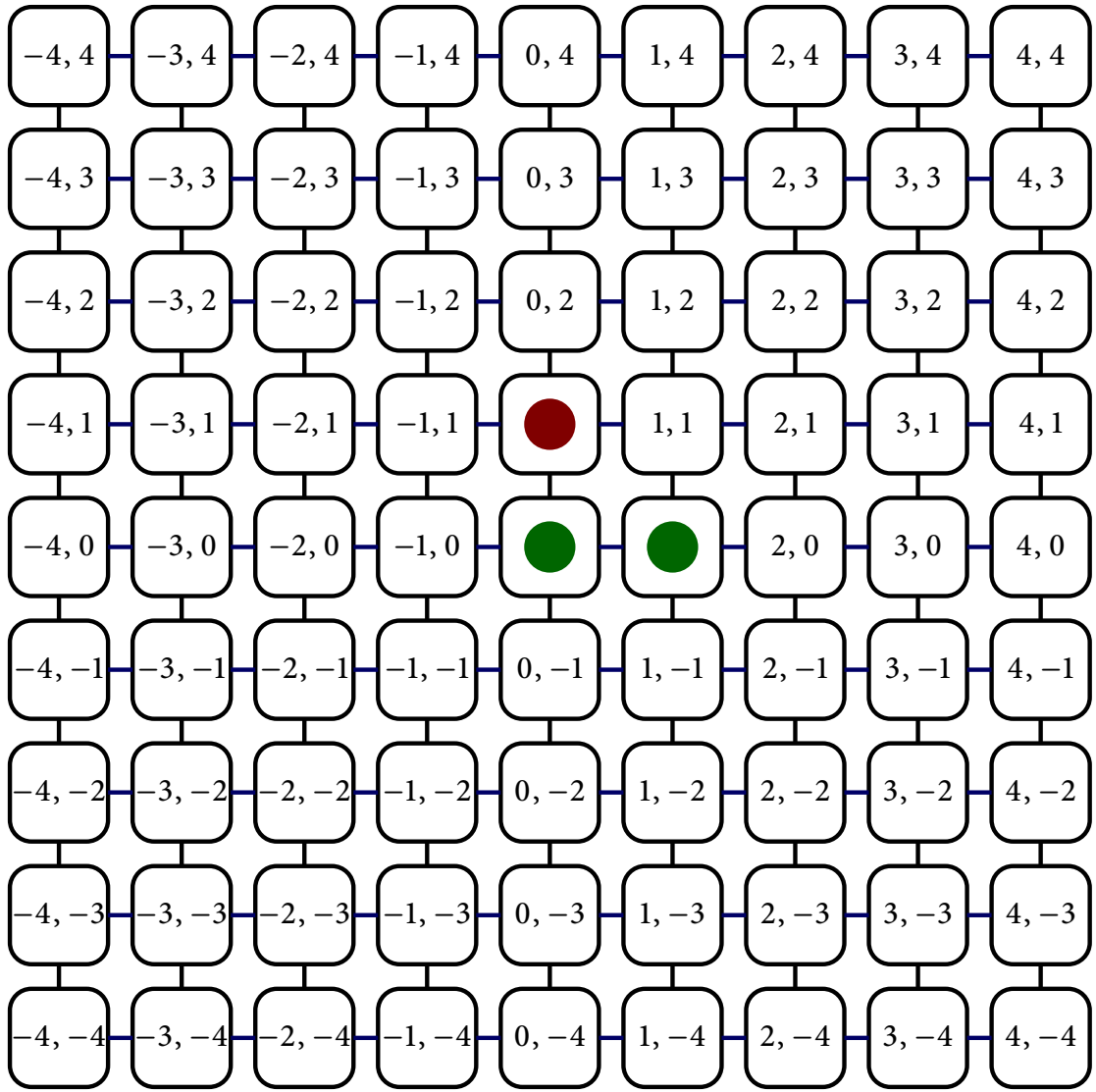

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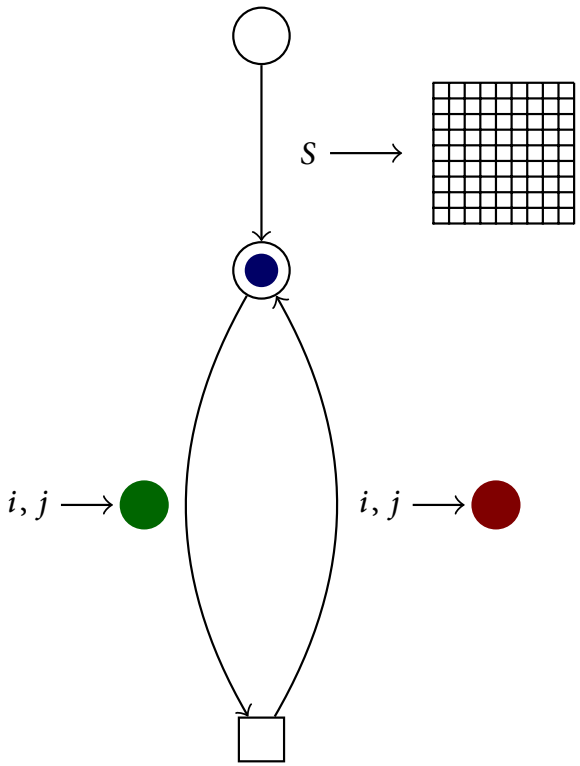
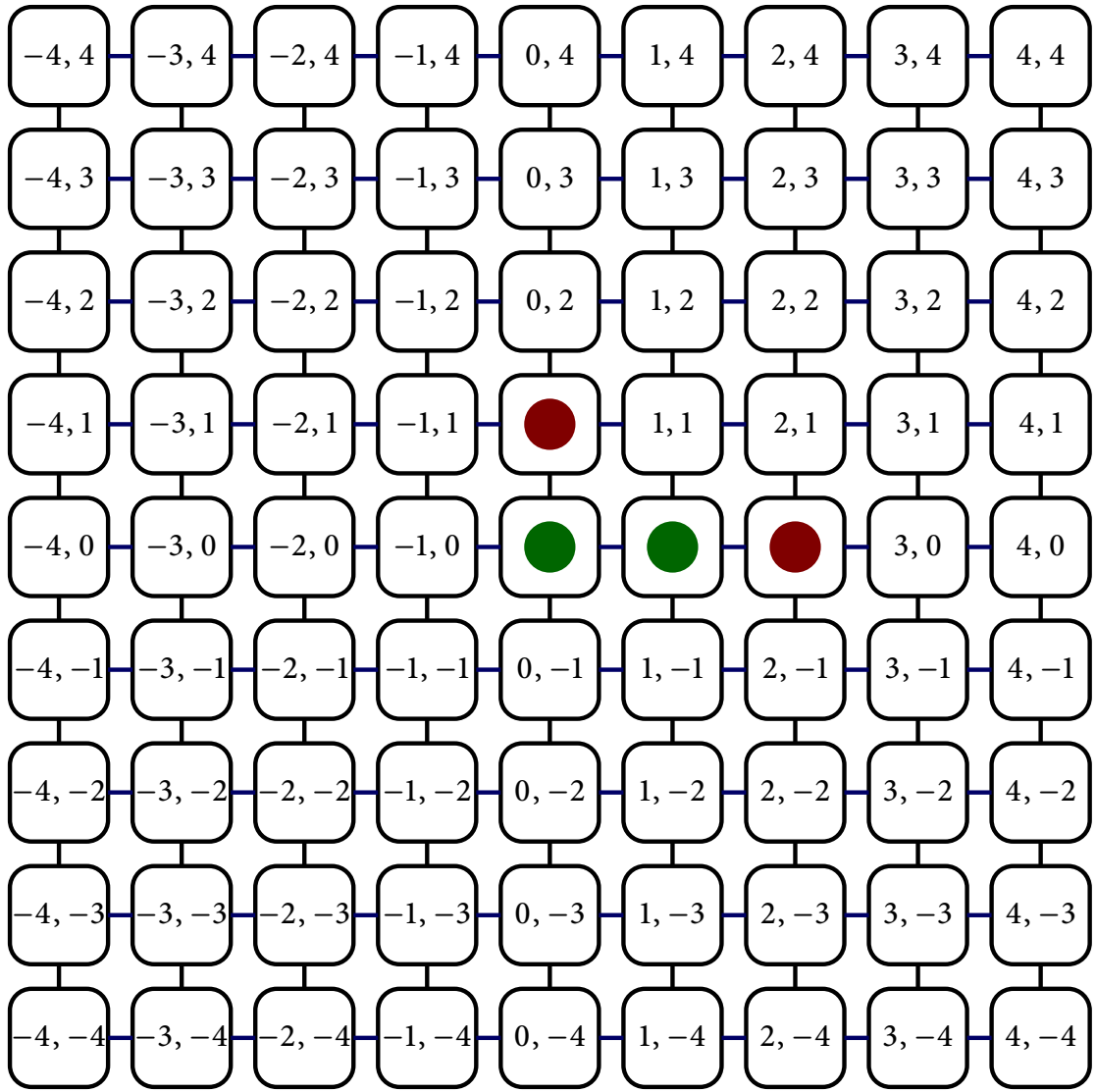
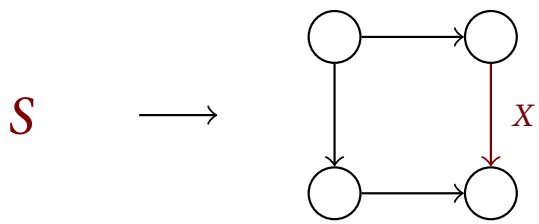




Example: Constructing a Ladder



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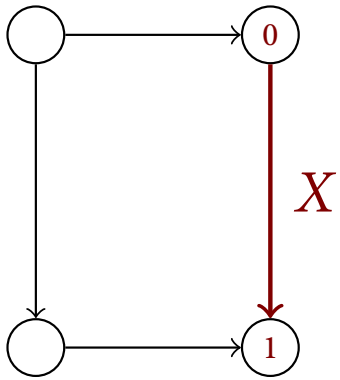

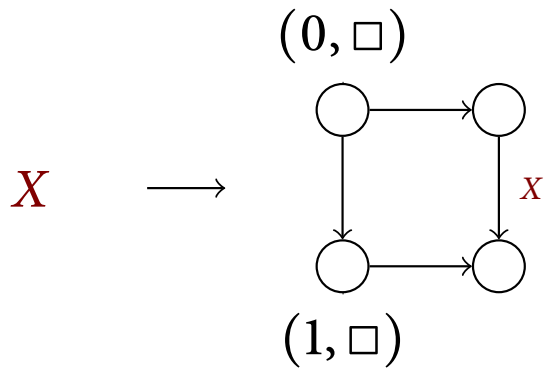

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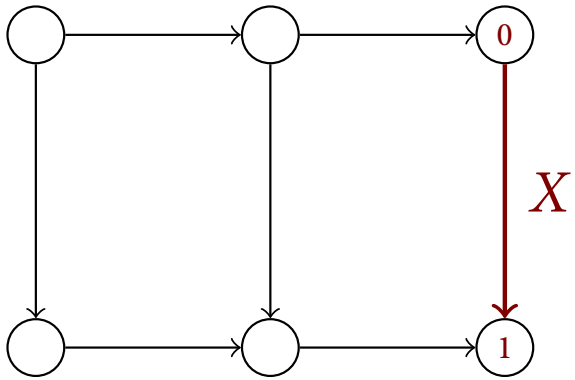

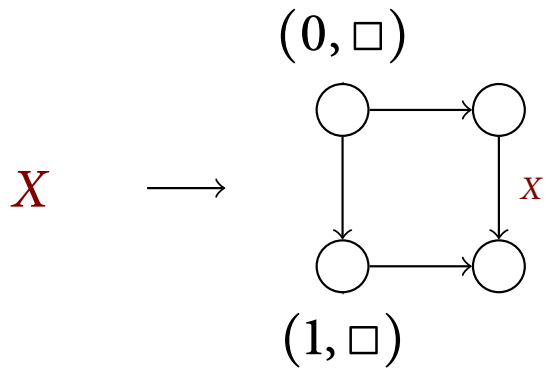

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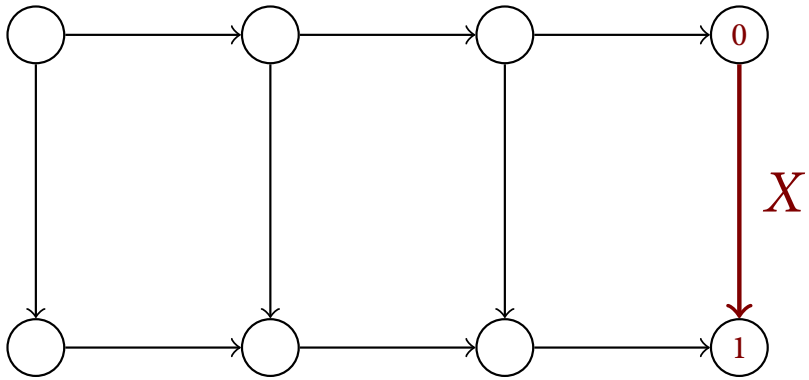

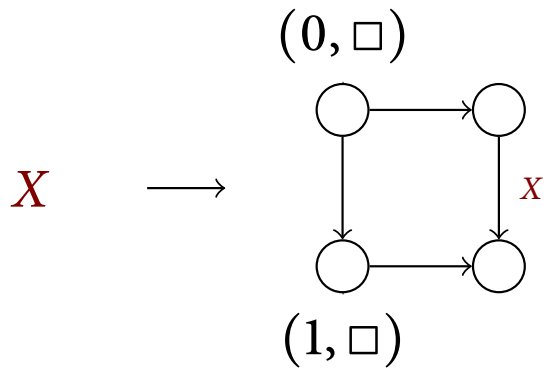

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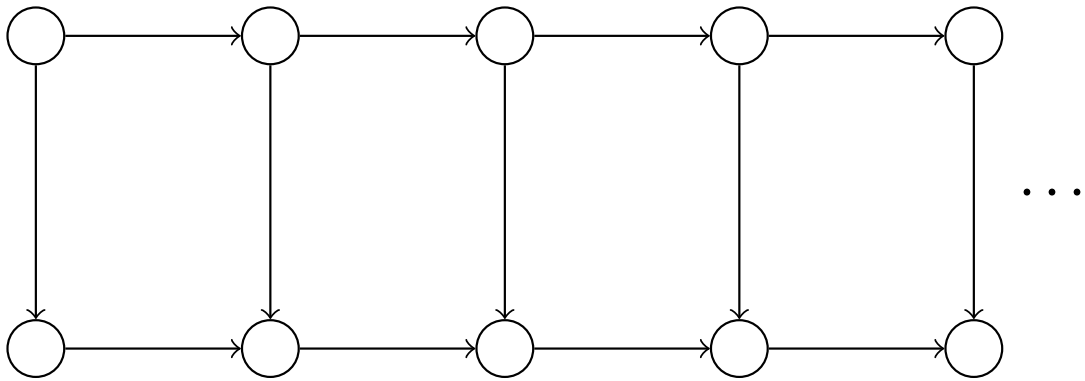


Main Result

Theorem

- *Let S be a **finite** set of **separated handle rewriting rules***
- *and φ be an MSO formula (giving the **winning condition**)*

*Then the set $\{\pi \in S^\omega \ : \ \lim G(\pi) \vDash \varphi\}$ is **ω-regular**.*

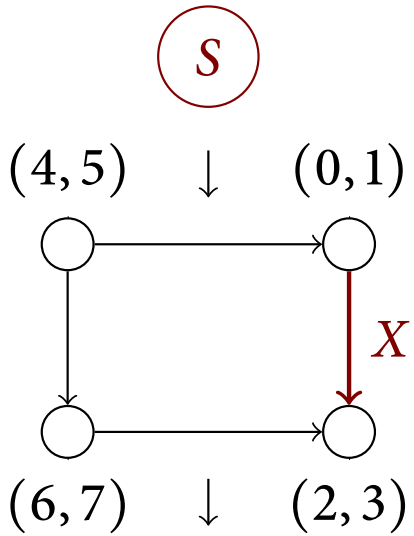
Corollary

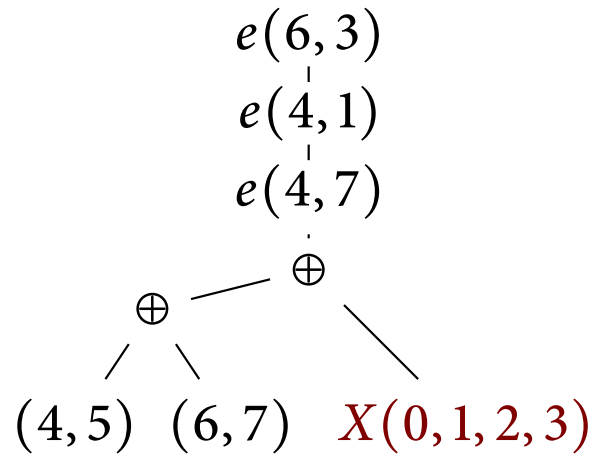
Establishing the winner of finite separated handle rewriting games is decidable.

Proof: Separated Graph Rewriting as a Tree

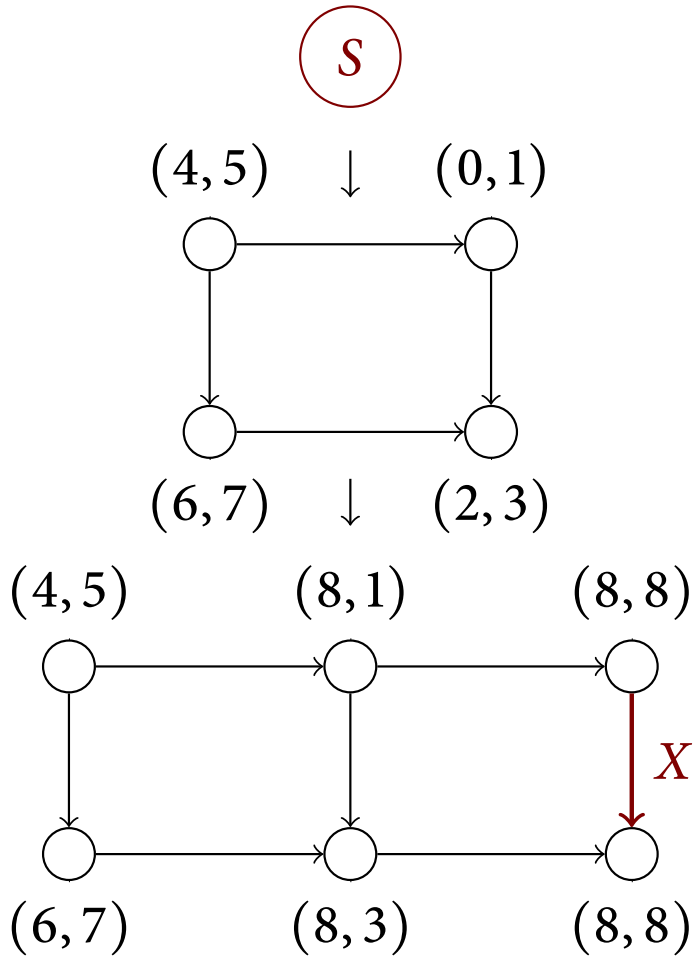
S

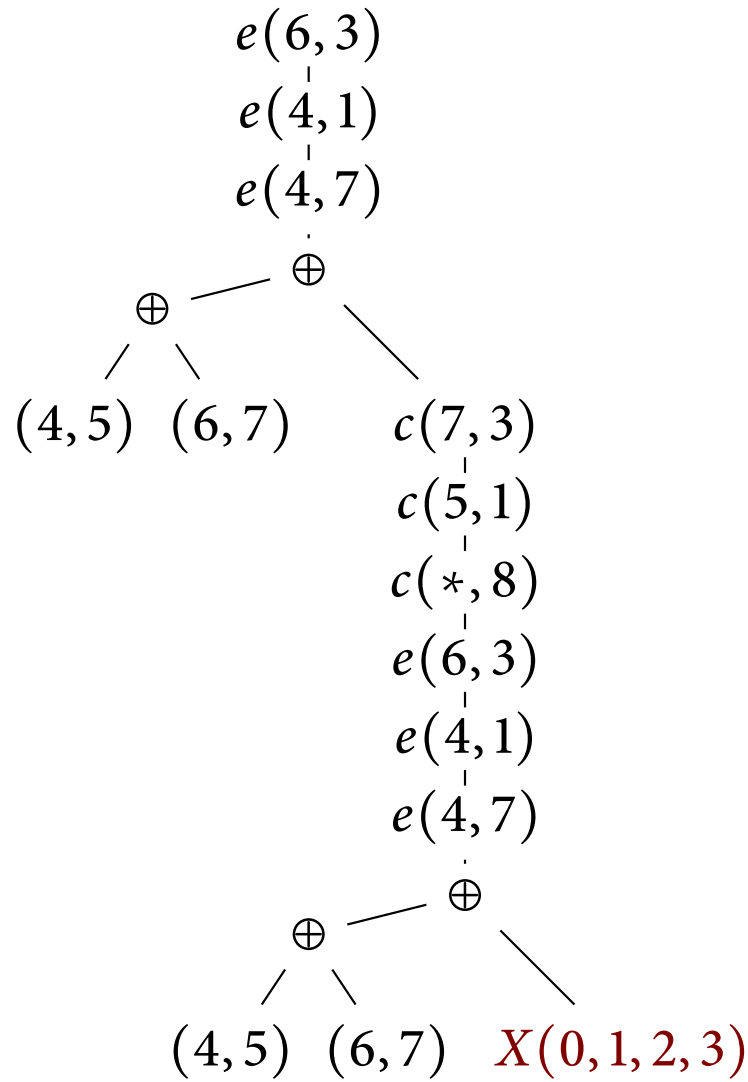
Proof: Separated Graph Rewriting as a Tree





Proof: Separated Graph Rewriting as a Tree





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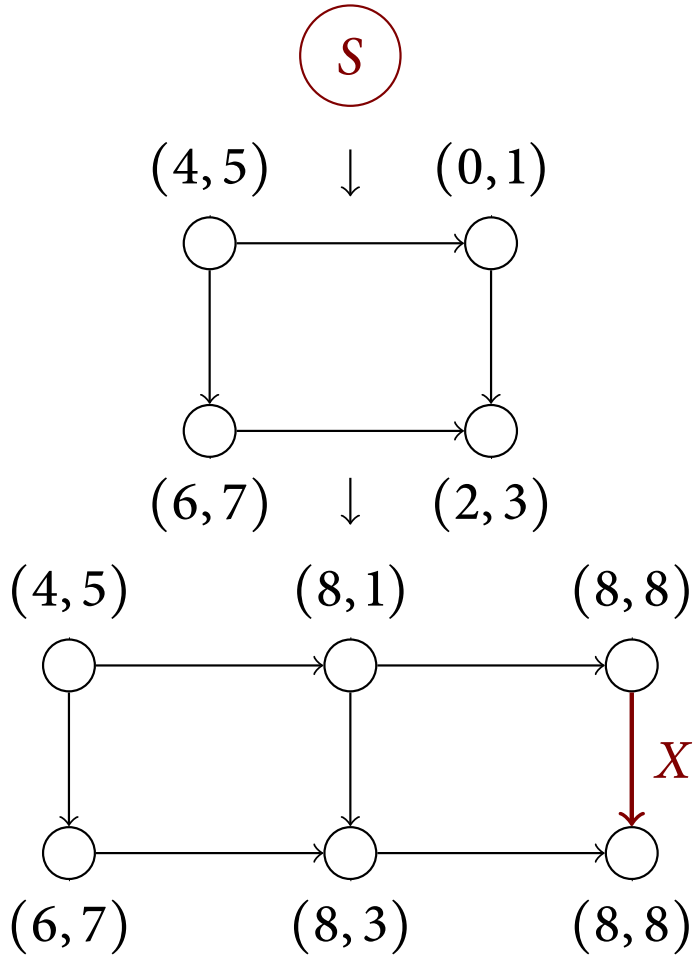
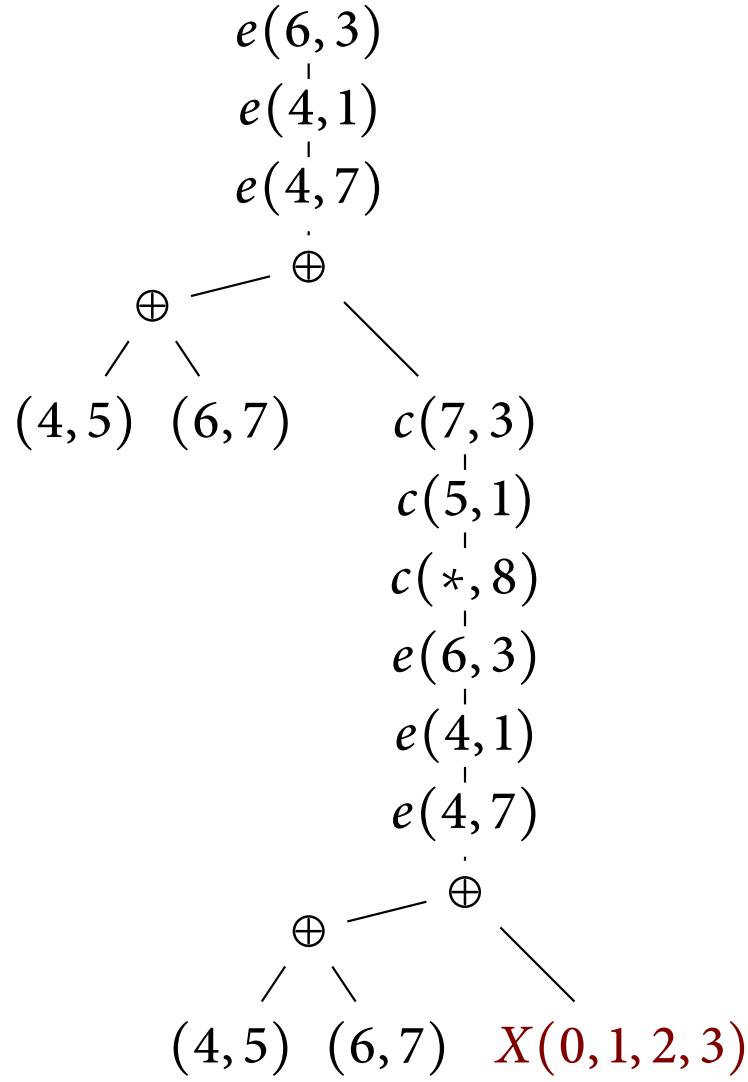
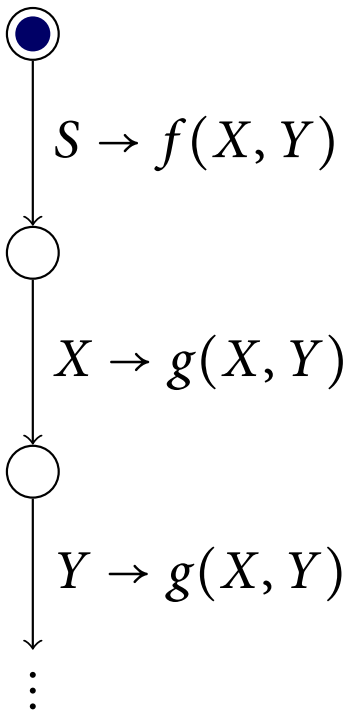




MSO-to-MSO **interpretation:** $\varphi \to \psi$

Proof: From Tree to Alternating Word Automata



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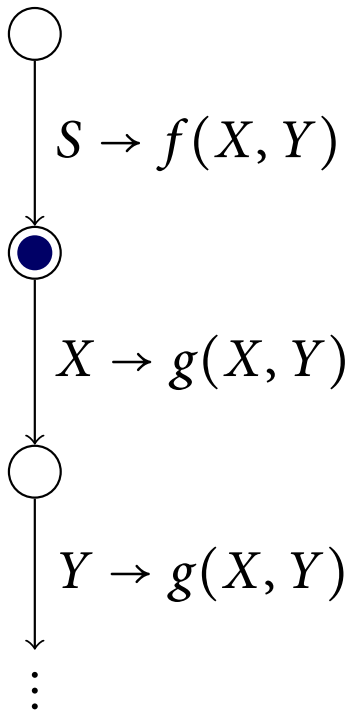

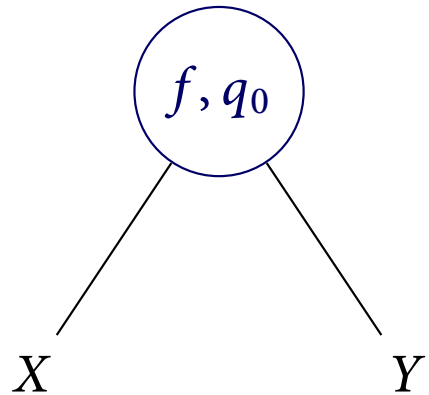

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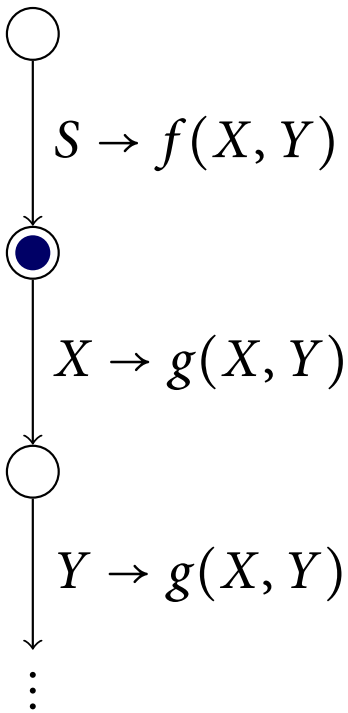

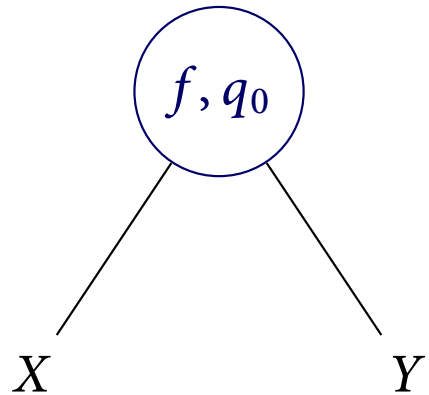

existential: pick transition

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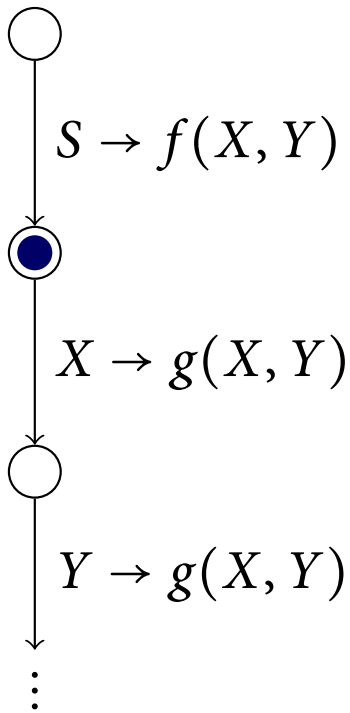

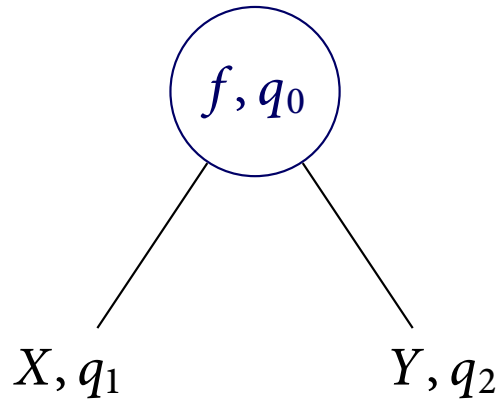

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$$f, q_0 \to (q_1, q_2)$$

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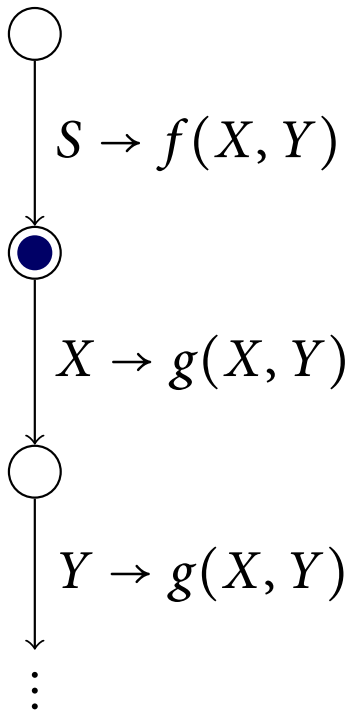

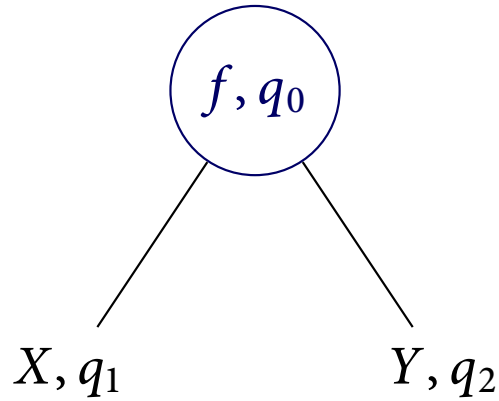

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universal: left or right

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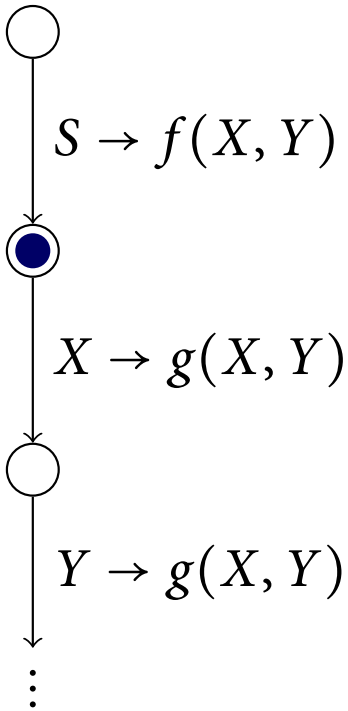

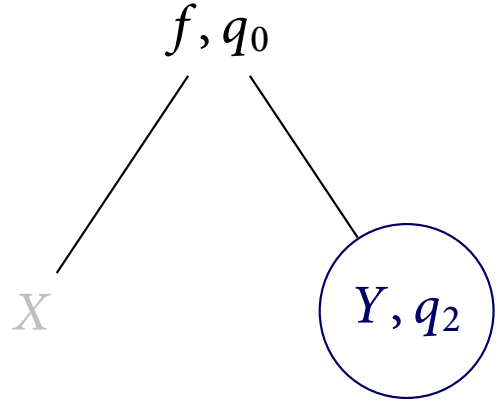

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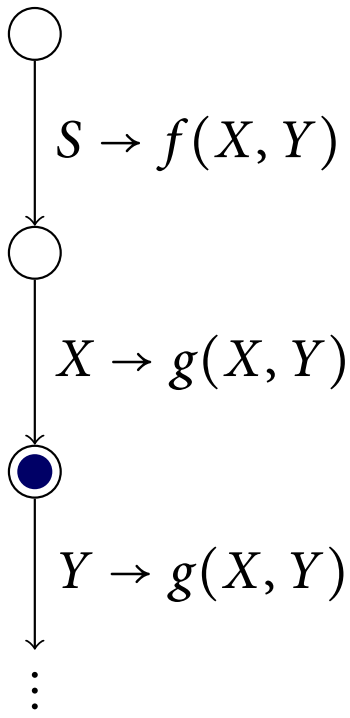

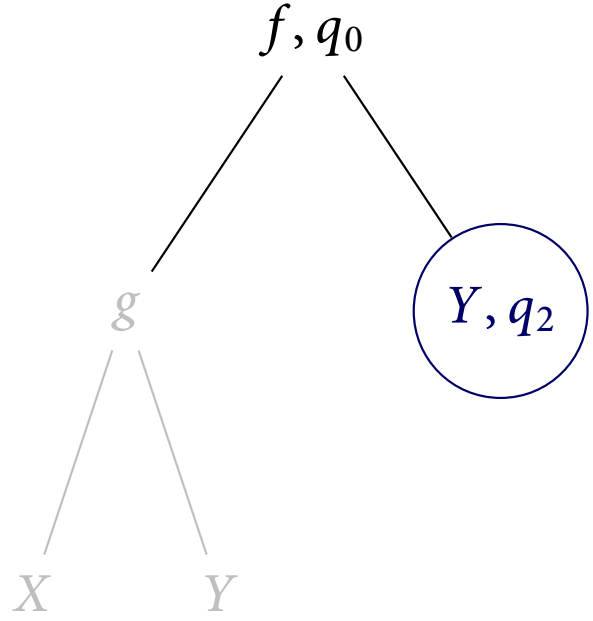

existential: pick transition

universal: left or right

$f, q_0 \to (q_1, q_2)$

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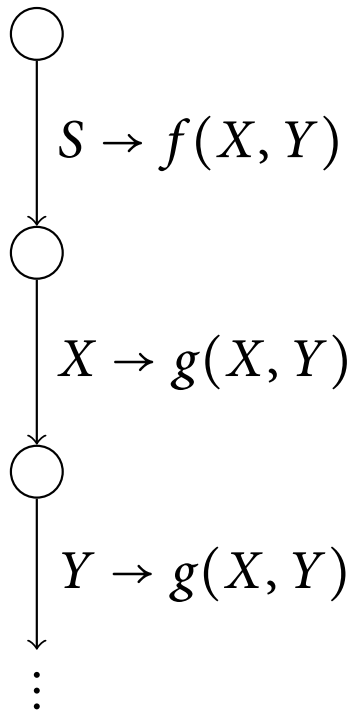
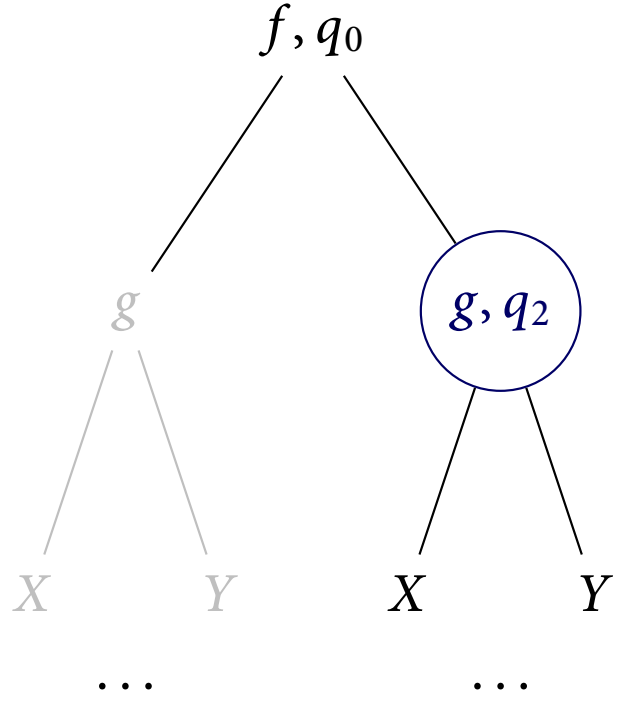
Proof: From Tree to Alternating Word Automata





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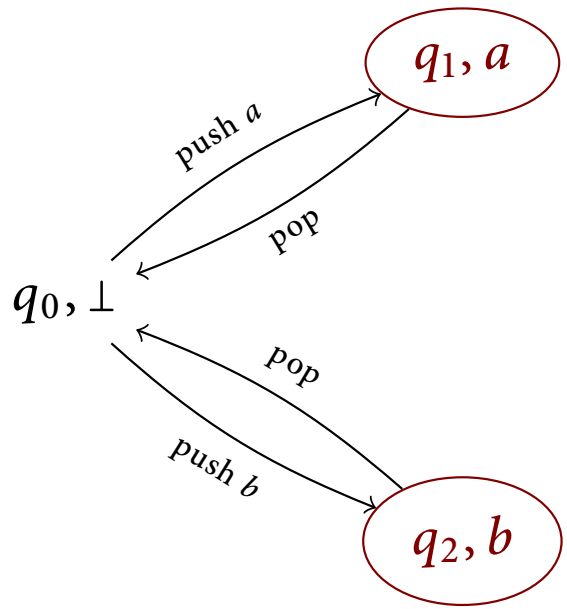

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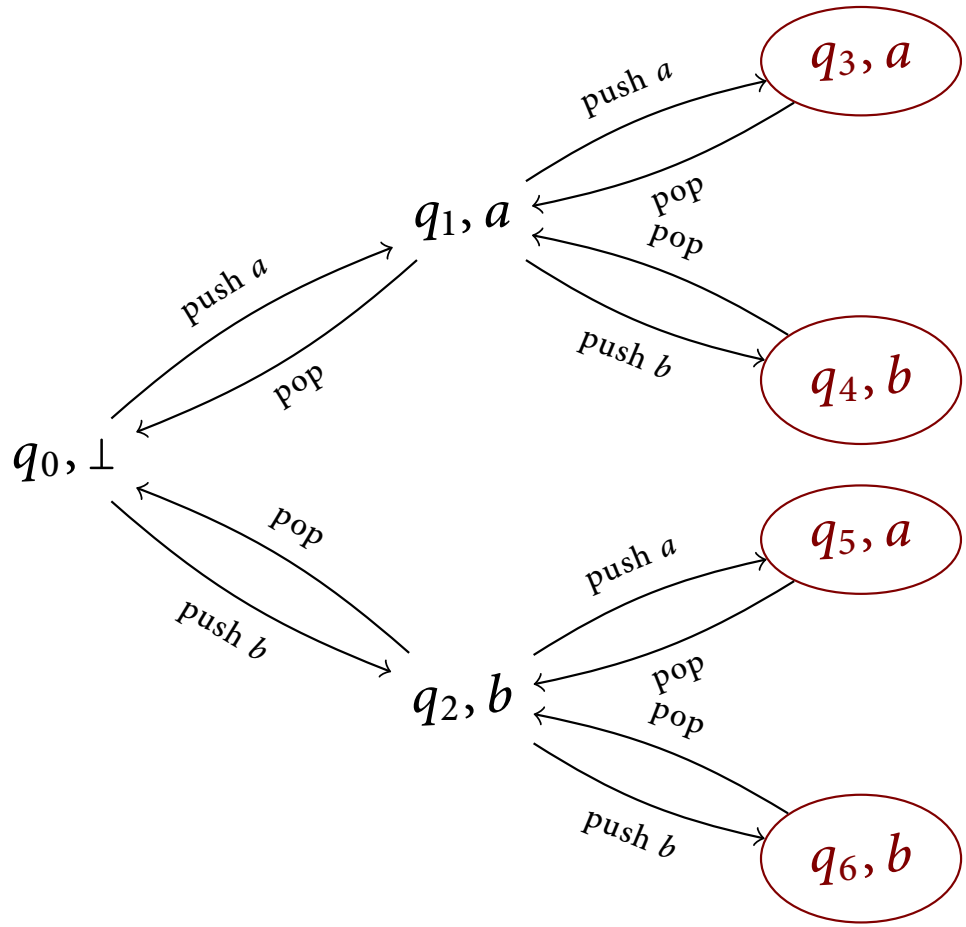

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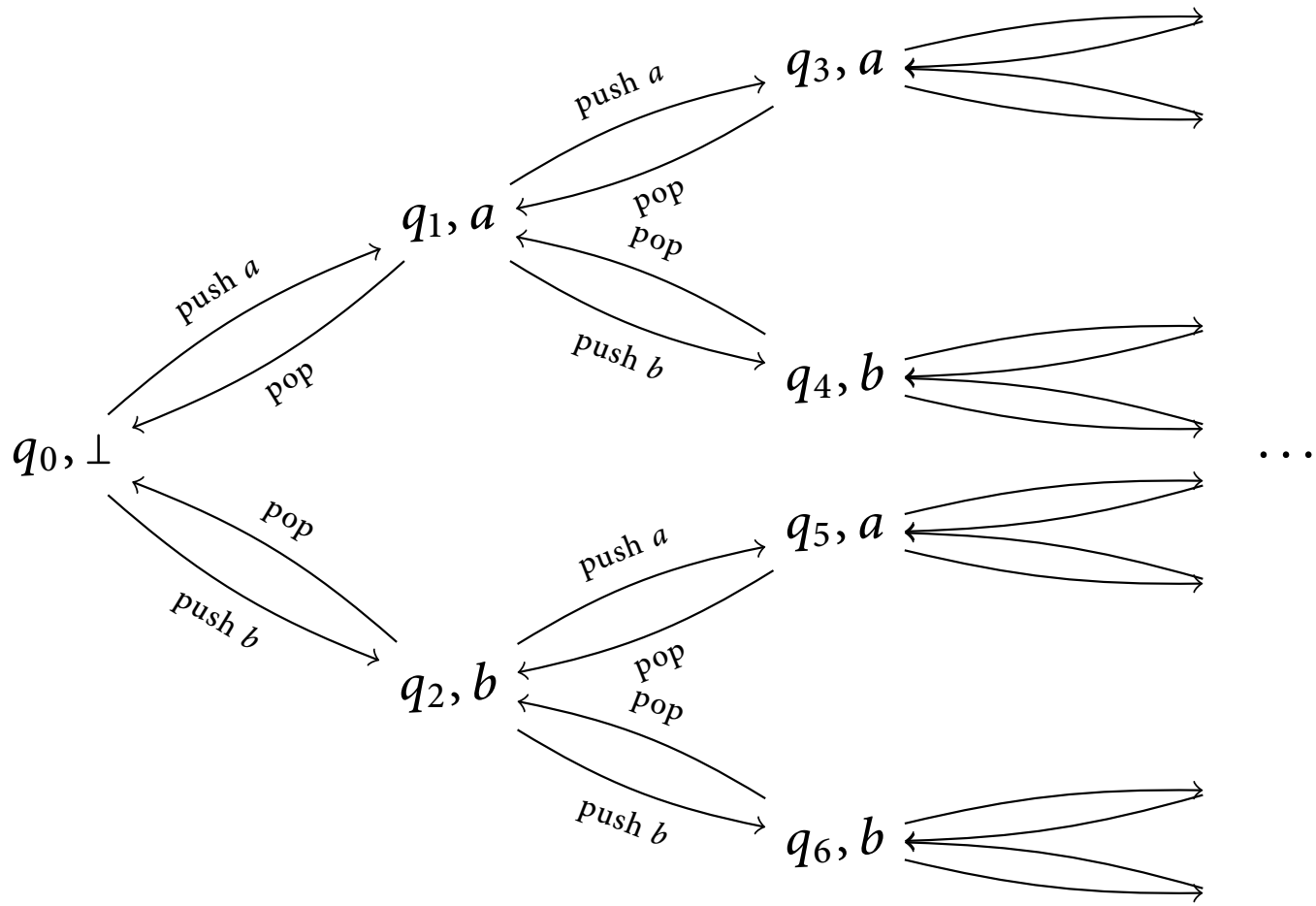

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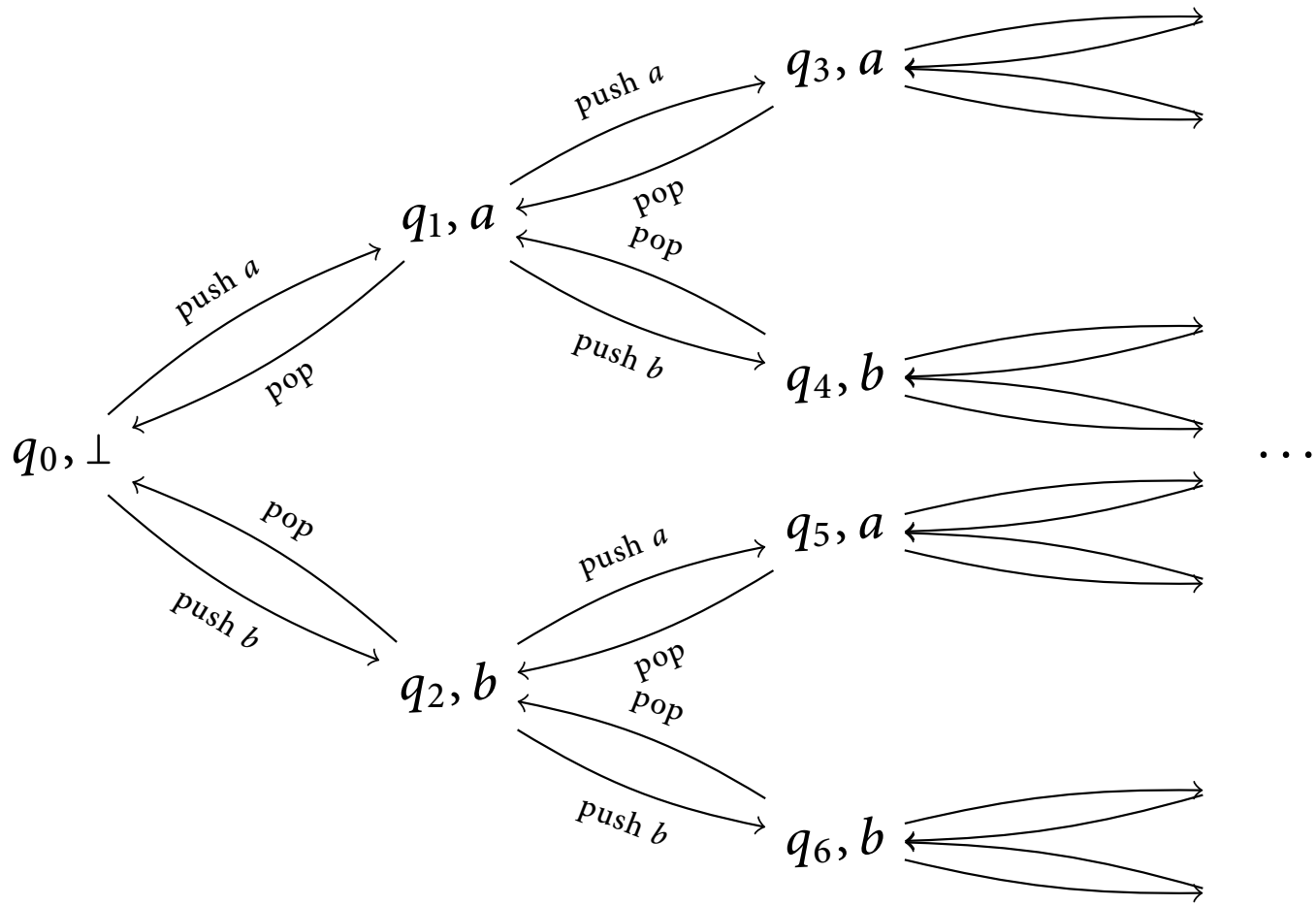

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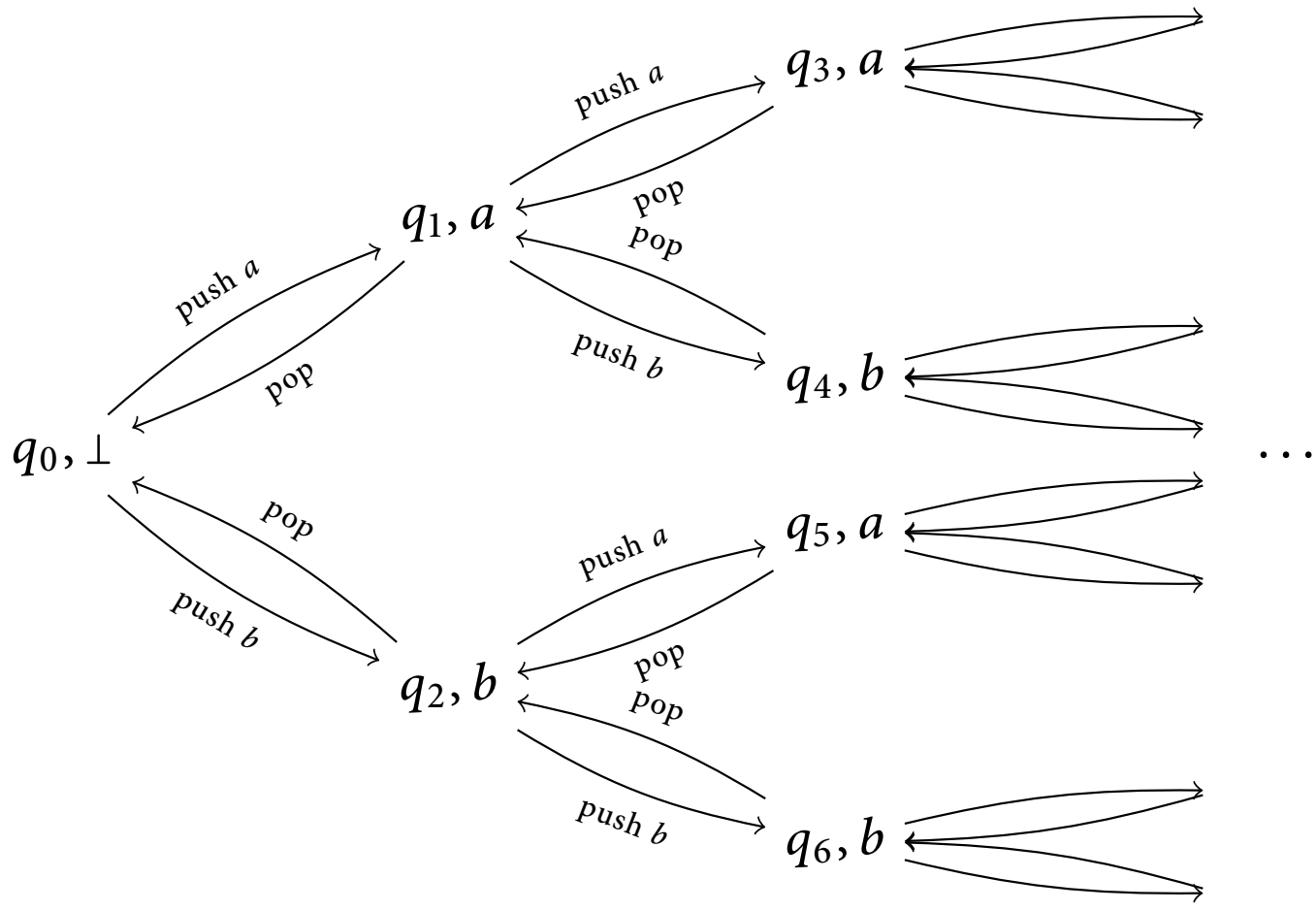

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Play a Game on the Resulting Graph

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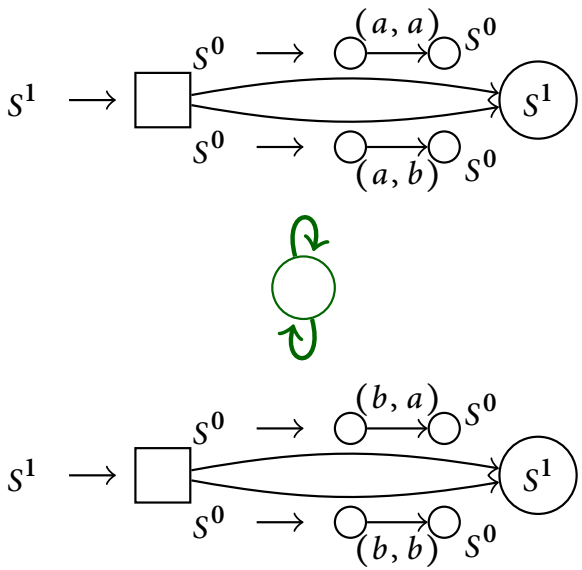
Higher-order game



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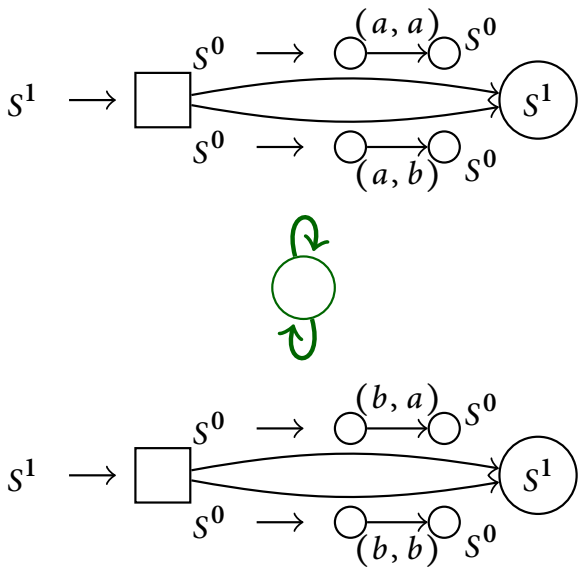

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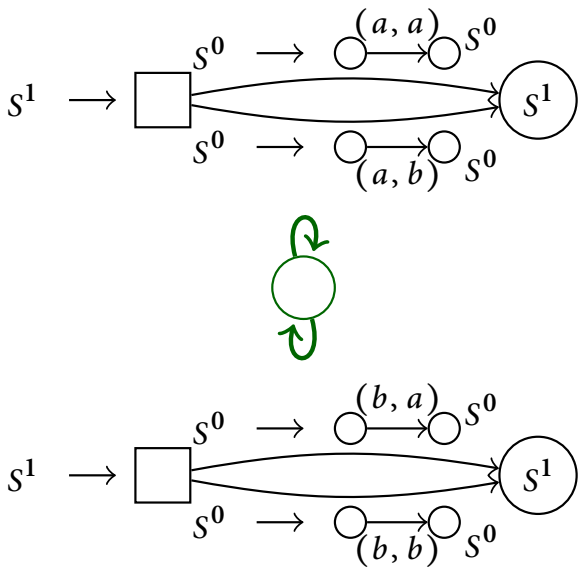

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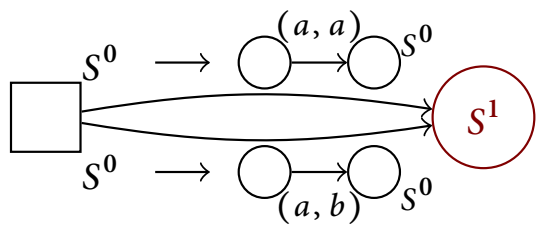

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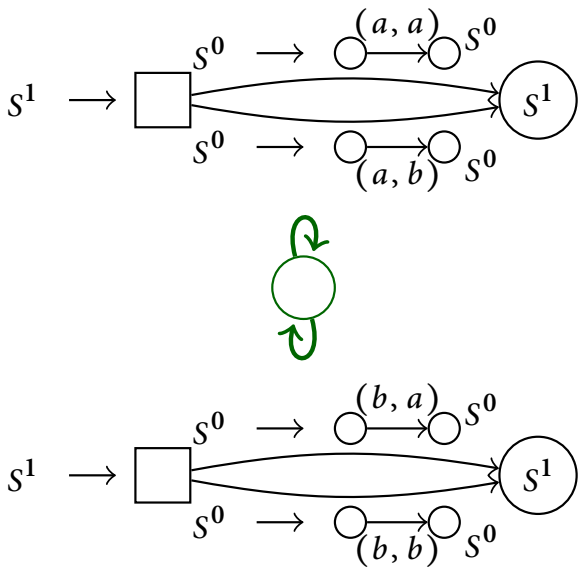

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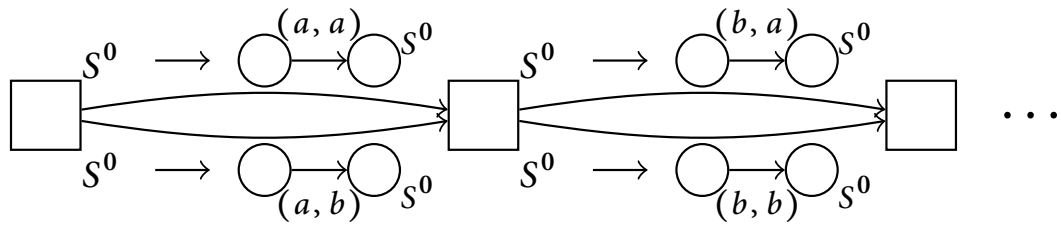

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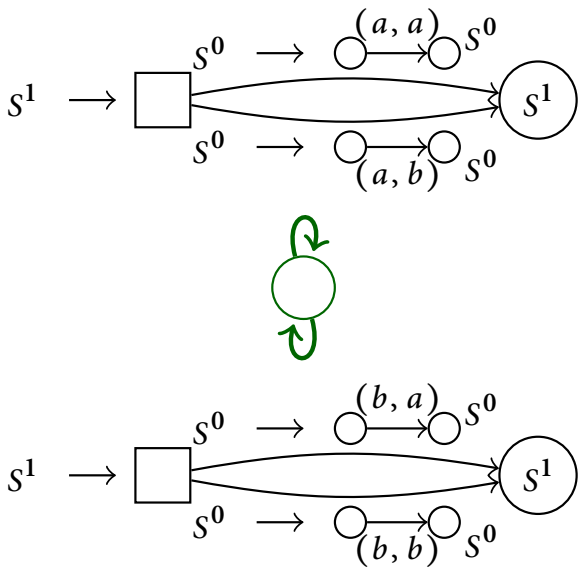

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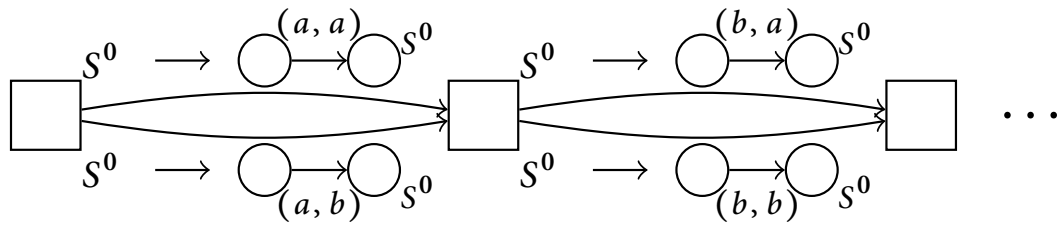

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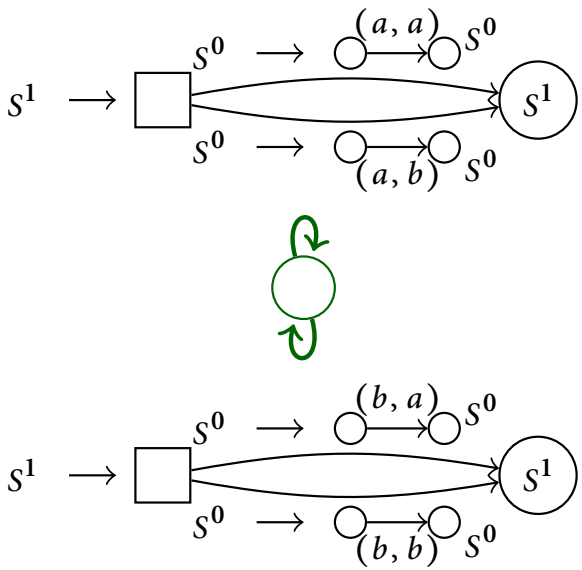

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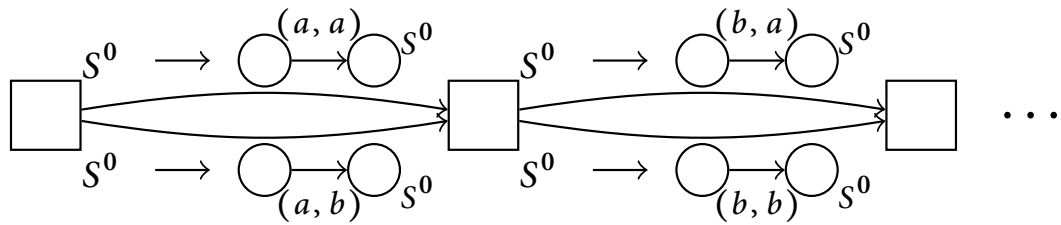

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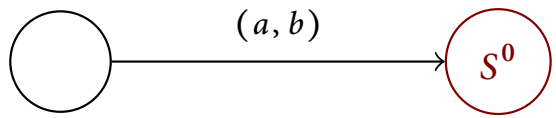

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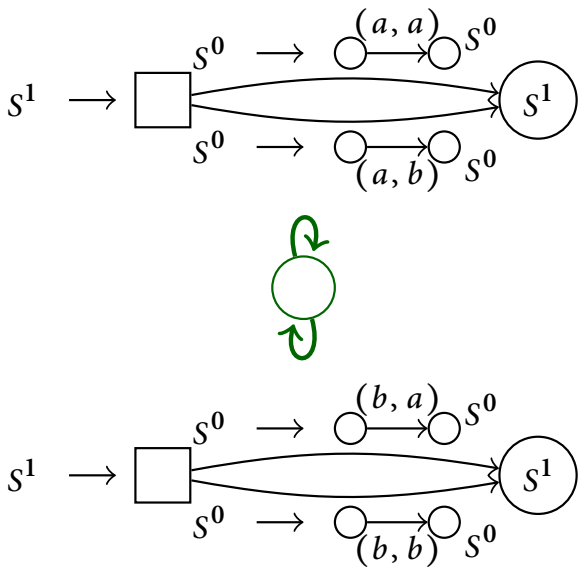

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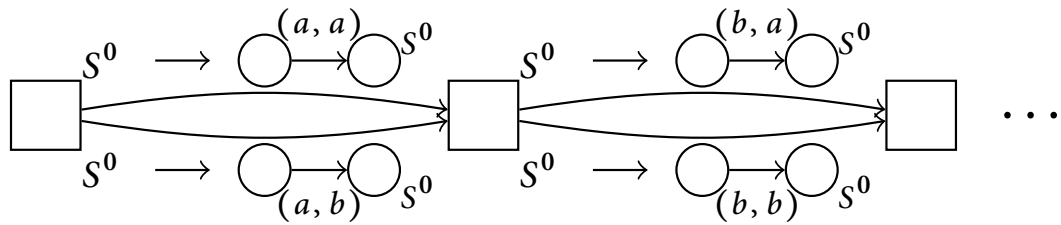

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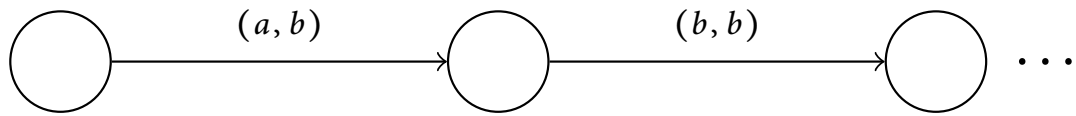

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Nice properties of separated handle rewriting games

- Reduce to **ω-regular games** for MSO winning conditions
- Establishing the winner is **decidable**
- Generate known **classes of graphs**, simulate **pushdown games**
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Thank You