Games with Structured States

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Mathematische Grundlagen der Informatik
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LIAFA Seminar

Paris, 2010

Overview

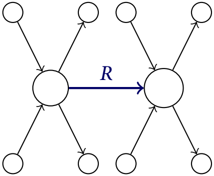
Structure Rewriting

Separated Games

Simulation-Based Playing

Structure Rewriting Rules

Rewriting Example



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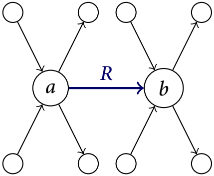

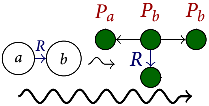

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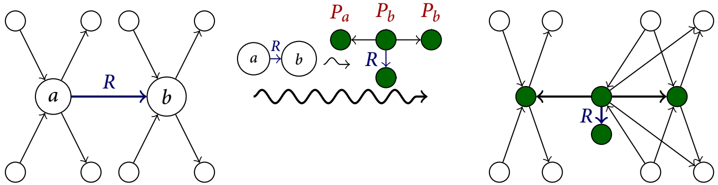

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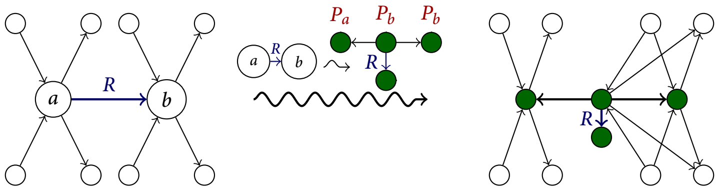

Relational Structures and Embeddings

$$\sigma \;:\quad \mathfrak{A} = (A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \ldots, R_k^{\mathfrak{A}}) \quad\rightarrow\quad (B, R_1^{\mathfrak{B}}, R_2^{\mathfrak{B}}, \ldots, R_k^{\mathfrak{B}}) = \mathfrak{B}$$

Embedding: σ is **injective** and $R_i^{\mathfrak{A}}(a_1, \ldots, a_{r_i}) \Leftrightarrow R_i^{\mathfrak{B}}(\sigma(a_1), \ldots, \sigma(a_{r_i}))$

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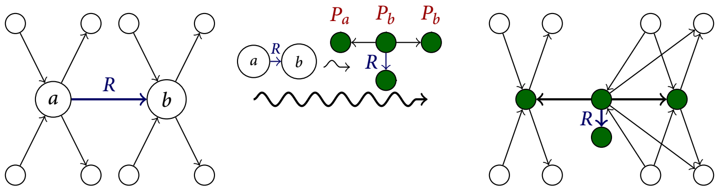

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Rewriting Definition

$\mathfrak{B} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$ iff $B = (A \setminus \sigma(L)) \dot\cup R$ and,
for $M = \{(r, a) \mid a = \sigma(l), r \in P_l^{\mathfrak{R}}$ for some $l \in L\} \cup \{(a, a) \mid a \in A\}$,

$$(b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{B}} \;\Leftrightarrow\; (b_1, \ldots, b_{r_i}) \in R_i^{\mathfrak{R}} \textbf{ or } (b_1 M \times \ldots \times b_{r_i} M) \cap R_i^{\mathfrak{A}} \neq \varnothing.$$

(in the second case at least one $b_j \notin \mathfrak{A}$)

Structure Rewriting Games

Game arena **(of a two-player zero-sum game)** is a **directed graph** with:

- vertices partitioned into positions of **Player 0** and **Player 1**
- edges **labelled by rewriting rules**

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- **Existential**: $\mathfrak{A}_{\text{next}} = \mathfrak{A}[\mathfrak{L} \to \mathfrak{R}/\sigma]$, the player chooses the embedding σ
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Winning conditions:

- L_μ (or temporal) formula ψ with **MSO sentences** for predicates, or
- MSO formula φ to be evaluated on the **limit** of the play
  Limit of $\mathfrak{A}_0\mathfrak{A}_1\mathfrak{A}_2 \ldots = \left(\bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} A_i,\ \bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} R^{\mathfrak{A}_i}\right)$
- **Reach** φ: Player 0 **wins** if the play reaches $\mathfrak{A}$ s.t. $\mathfrak{A} \models \varphi$

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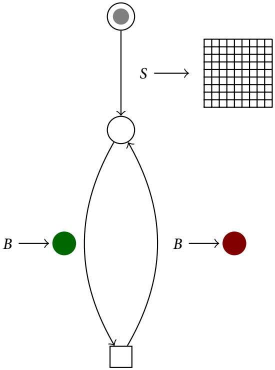
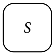
Motivation: many questions are **naturally defined as such games**:
constraint satisfaction, model checking, graph measures, games people play

Example Game: Gomoku (Connect–5)





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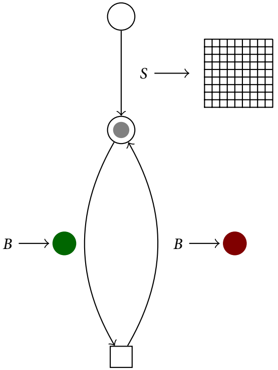

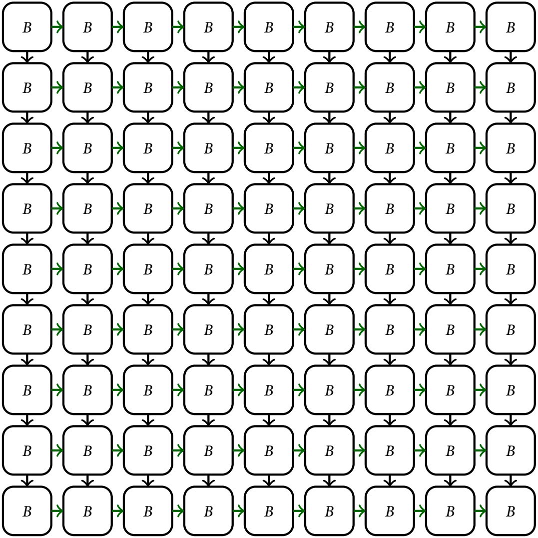

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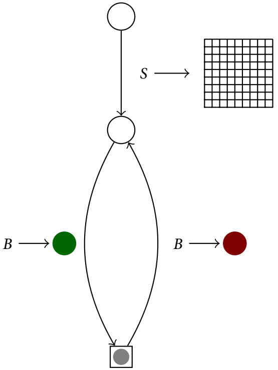

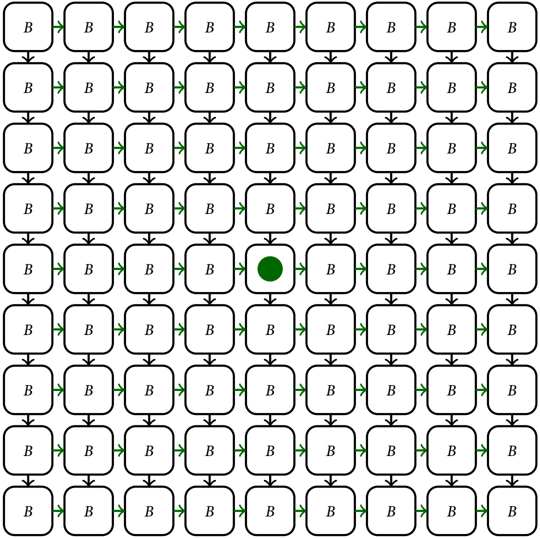

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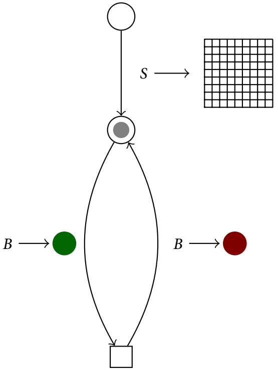

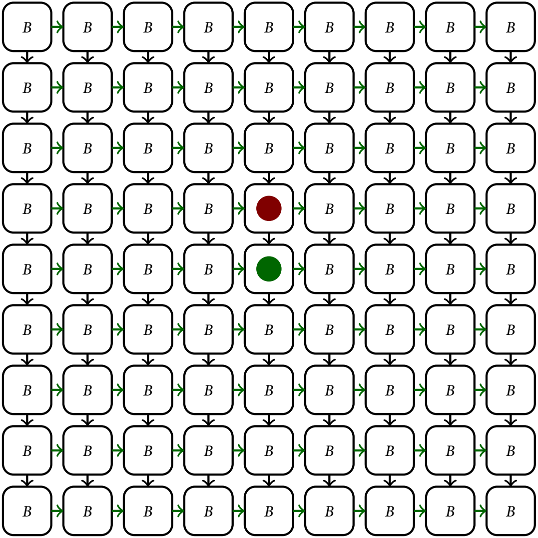

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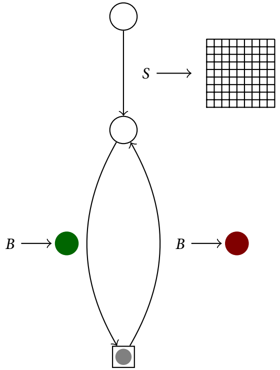

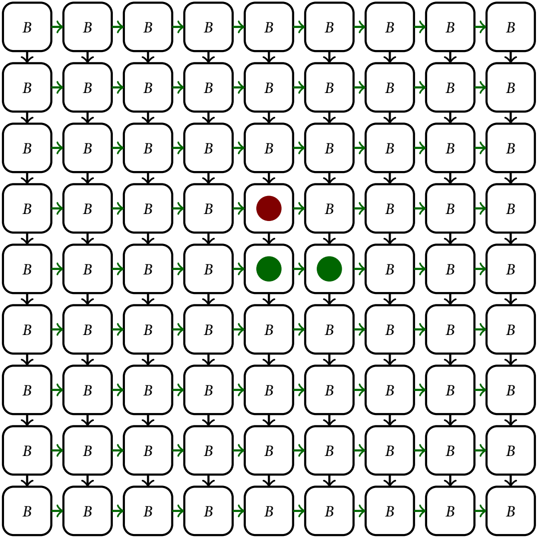

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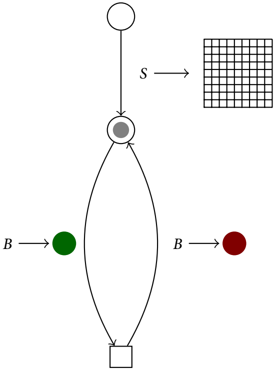

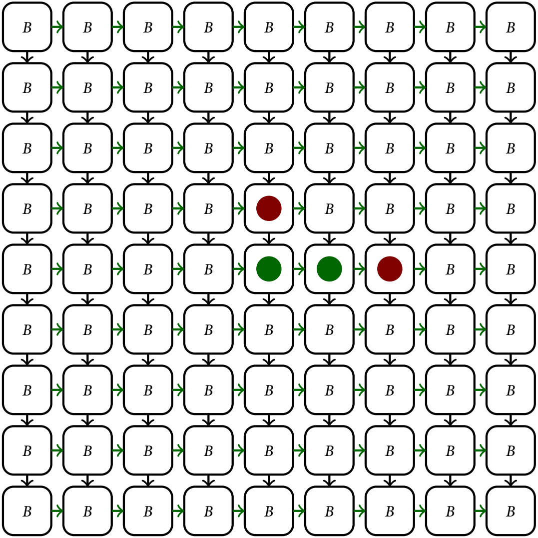

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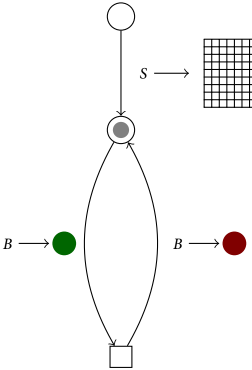

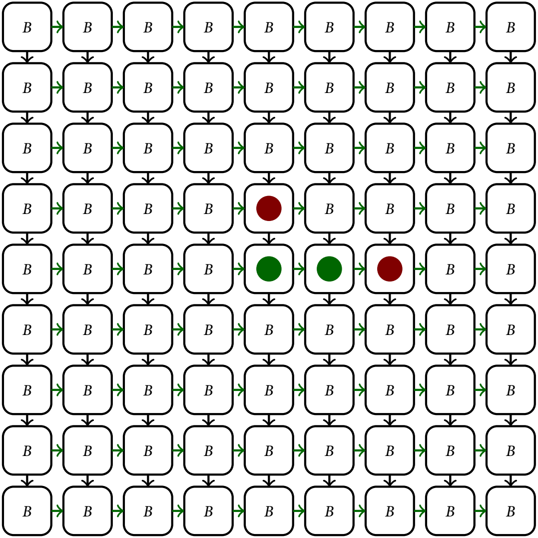

$$\exists x_1 \dots x_5 \Big(\bigwedge_{1\le i\le 5} G(x_i) \wedge \Big(\bigwedge_{1\le i\le 5} R(x_i, x_{i+1}) \vee \bigwedge_{1\le i\le 5} C(x_i, x_{i+1}) \vee \bigwedge_{1\le i\le 5} \exists y (R(x_i, y) \wedge C(y, x_{i+1})) \vee \bigwedge_{1\le i\le 5} \exists y (R(x_i, y) \wedge C(x_{i+1}, y)) \Big) \Big)$$

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Simple Structure Rewriting

Separated Structures: no element is in two **non-terminal** relations
(Courcelle, Engelfriet, Rozenberg, 1991)

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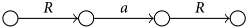

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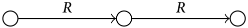

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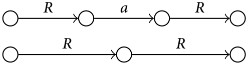

Simple Rule $\mathfrak{L} \to \mathfrak{R}$: $\mathfrak{R}$ is **separated** and $\mathfrak{L}$ is a **single tuple in relation**

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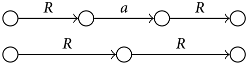

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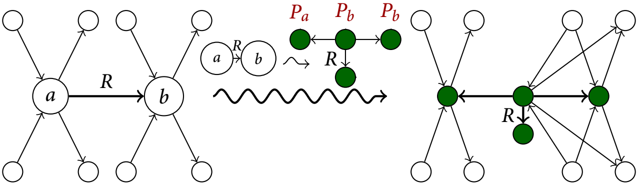

Decidability of Simple Rewriting Games

Logics

- L_μ[MSO]: Temporal properties expressed in L_μ (subsumes LTL) with properties of structures (states) expressed in MSO
- lim MSO: Property of the limit structure expressed in MSO

Theorem

- *Let R be a* ***finite*** *set of* ***(universal) simple structure rewriting rules,***
- *and φ be an L_μ[MSO] or lim MSO formula.*

Then the set $\{\pi \in R^\omega \ : \ (\lim)S(\pi) \vDash \varphi\}$ is ***ω-regular****.*

Corollary

Establishing the winner of (universal) finite simple rewriting games is decidable.
The winner has a winning strategy of a simple form.

Proof: Interpreting a Structure in a Tree

Description of how to build $\mathfrak{A}$ is a tree $\mathcal{T}(\mathfrak{A})$ with:

- **Leafs** of **different colours** $1 \dots k$
- $\oplus$ representing **disjoint sum**
- $i \leftarrow j$ to **change colour** of **all** nodes from i to j
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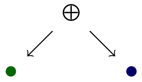

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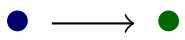

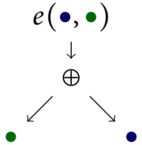

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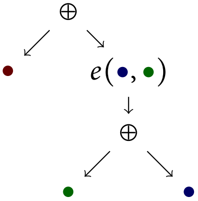

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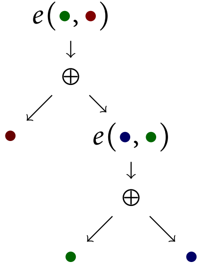

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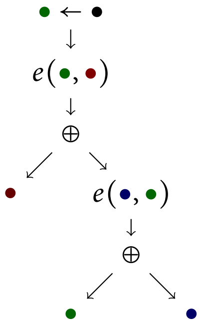

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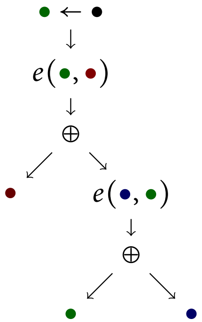

Theorem:
For every k there is an MSO-to-MSO **interpretation** $\mathcal{I}$ such that for all structures $\mathfrak{A}$ of **clique-width** $\leq k$ holds $\mathcal{I}(\mathcal{T}(\mathfrak{A})) \cong \mathfrak{A}$.

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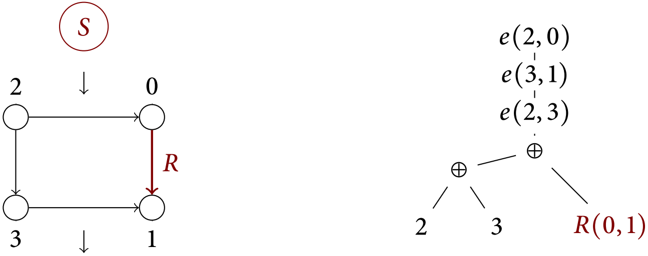

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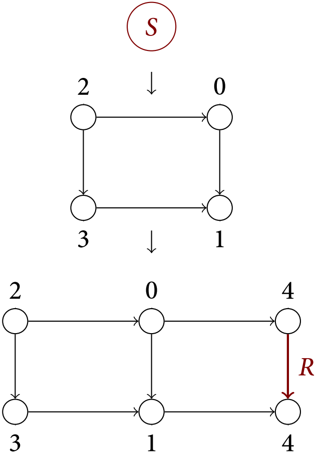

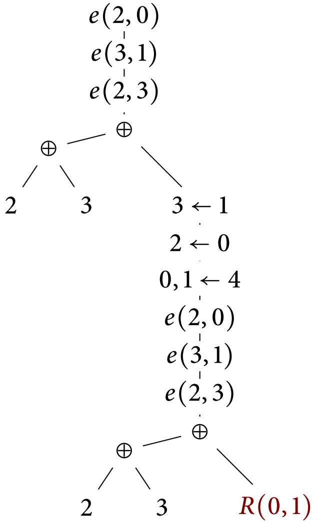

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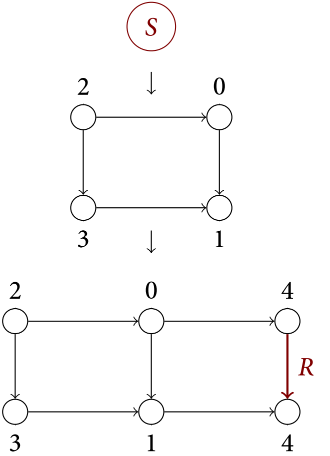
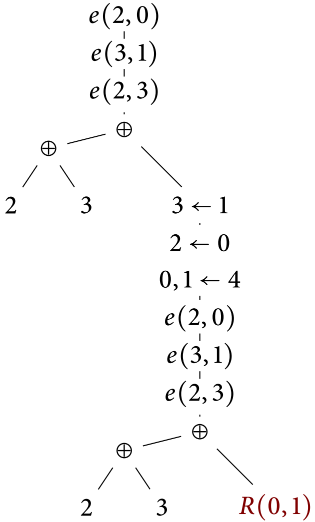
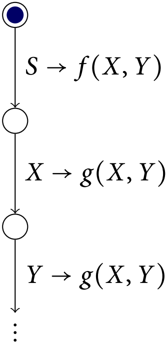




MSO-to-MSO interpretation: $\varphi \to \psi$

Proof: From Tree to Alternating Word Automata



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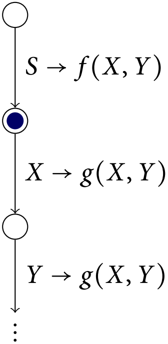

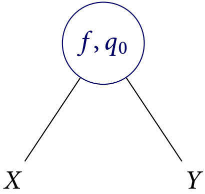

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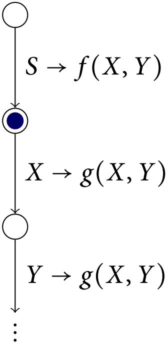

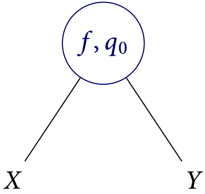

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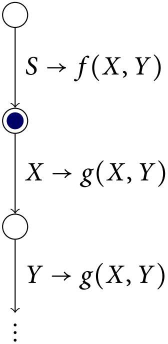

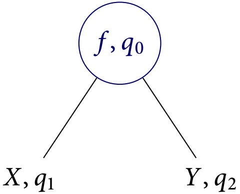

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$f, q_0 \to (q_1, q_2)$

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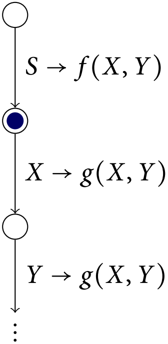

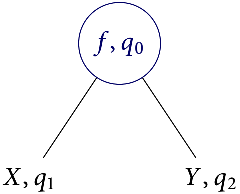

existential: pick transition

universal: left or right

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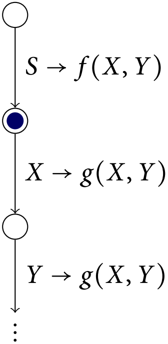

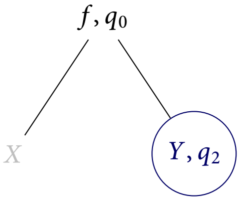

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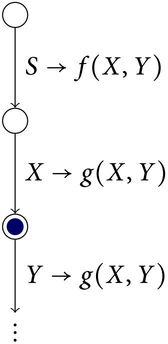

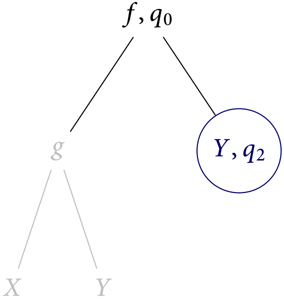

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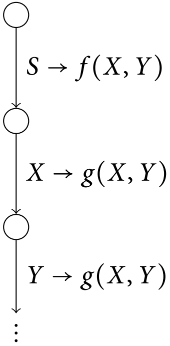

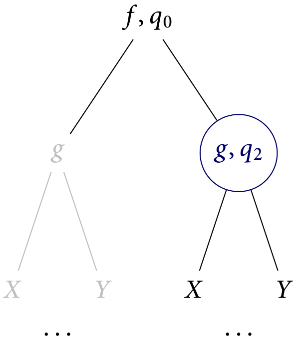

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Overview

Structure Rewriting

Separated Games

Simulation-Based Playing

Motivation

Implementing the Decidability Result

- **Tool**: **MONA**
  - Developed at **BRICS** since **1996** by Nils Klarlund and Anders Møller
  - Symbolic representation with **BDDs**
  - **Minimisation** at each step
- **Example:** a simple **tic-tac-toe** game $\rightsquigarrow$ **memory overflow**
- **Problems** due to **inefficient coding**
  - **Bounded clique-width graphs** not good for MONA
  - Only **universal interpretation** decidable, must encode games

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Building a Tree during Random Plays

- **Idea:** memorise first random moves, play **minimax** there
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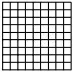

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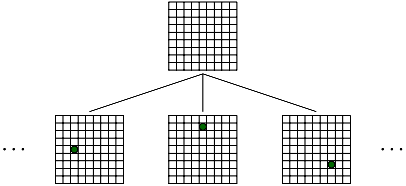

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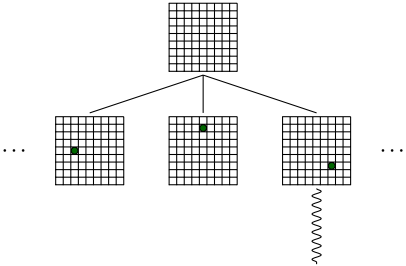

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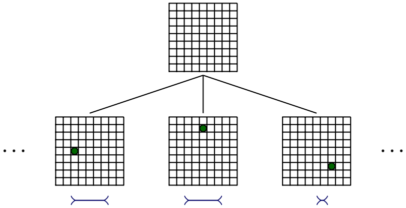
$$C \cdot \sqrt{\frac{\ln(n(v)+1)}{n(w)+1}}$$



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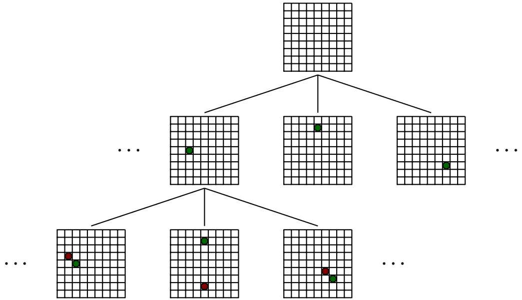

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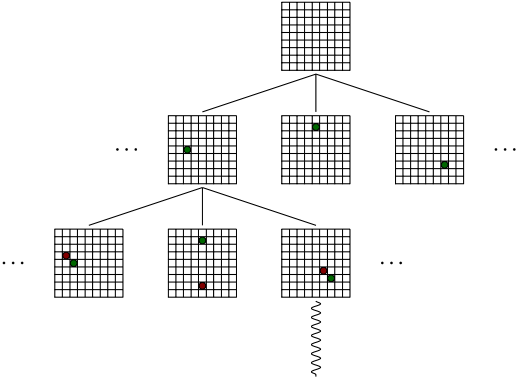

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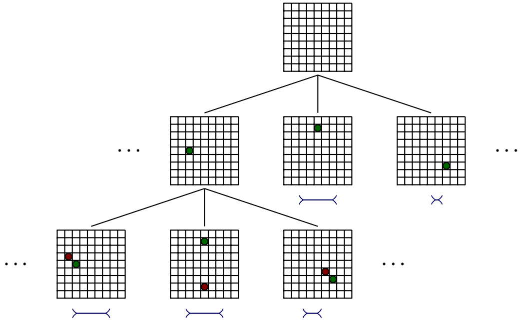

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- Fully random player is **too general**
- Large number of **formula evaluations** (slow)

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Perspective: once a good hint is found, prove that the strategy is **winning**.

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Solver Requirements

(1) **Obvious**: evaluate formulas fast

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Current Solver Architecture (`toss.sourceforge.net`)

- **FO assignments**: represented directly
- **MSO assignments**: semi-symbolically

$$(1 \in X \wedge 2 \in X \wedge 3 \notin X) \vee (1 \notin X)$$

- Operations on MSO assignments: **use SAT solver**, **CNF-DNF again**
- Are **BDDs** better? **Unclear.**

Conclusions

Structure Rewriting Games

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